Published by Institute of Physics Publishing for SISSA

Received: *April 10, 2006*
Revised: *June 29, 2006*
Accepted: *July 4, 2006*
Published: *August 9, 2006*

Parent form for higher spin fields on anti-de Sitter space

Glenn Barnich*

Physique Théorique et Mathématique, Université Libre de Bruxelles and International Solvay Institutes,
Campus Plaine C.P. 231, B-1050 Bruxelles, Belgium
E-mail: gbarnich@ulb.ac.be

Maxim Grigoriev

Tamm Theory Department, Lebedev Physics Institute,
Leninsky prospect 53, 119991 Moscow, Russia
E-mail: grig@td.lpi.ac.ru

Abstract: We construct a first order parent field theory for free higher spin gauge fields on constant curvature spaces. As in the previously considered flat case, both the original formulation by Fronsdal and the unfolded one by Vasiliev can be reached by two different straightforward reductions. The parent theory itself is formulated using a higher dimensional embedding space. It turns out to be geometrically extremely transparent and free of the intricacies of both of its reductions.

Keywords: String Field Theory, BRST Quantization, Models of Quantum Gravity.

*Senior Research Associate of the National Fund for Scientific Research (Belgium).

Contents

1. Introduction

Progress in the subject of higher spin gauge fields has often been related with the construction of new equivalent formulations of the theory. A Lagrangian formulation at the free level [1 – 4] required the introduction of a carefully selected set of auxiliary fields. Other auxiliary fields were needed for the BRST reformulation, directly inspired by string field theory (see e.g [5 – 13] and references therein), in terms of the field theory associated to a BRST first quantized particle model [14 – 16] (see e.g. [17] for a review). This reformulation explicitly revealed the relation with the tensionless limit of string theory and provided a compact Lagrangian description at the free level. Similarly, finding a consistent interaction [18, 19] on an anti-de Sitter (AdS) background was done exclusively in the *unfolded formulation* [20, 21] that also provides a natural framework for various other problems of higher spin theories [22 – 24].

Further developments, such as for example understanding whether the recently constructed interaction in an AdS background admits a Lagrangian formulation, require a good control over various equivalent formulations differing by auxiliary and pure gauge fields. As a first step, one would like to explicitly relate the unfolded formulation and the BRST formulation. In the case of a flat background, this has been done recently through the construction of a parent theory [25] from which both formulations can be reached through consistent reductions. Furthermore, some algebraic structures that are hidden in the BRST or unfolded formulations appear more transparently in the parent theory or some of its intermediate reductions. The objective of the present paper is to extend these results to an AdS background.

From a more technical point of view, understanding free higher spin gauge fields in terms of a first-quantized generally covariant particle model has two advantages: firstly, it allows one to transpose the arsenal of cohomological methods available at the BRST first quantized level to the gauge field theory. In particular, because auxiliary fields and pure gauge degrees of freedom can be identified with cohomologically trivial pairs at the first-quantized level, showing the equivalence of various formulations boils down to a straightforward exercise in homological algebra. Secondly, well known quantization techniques for complicated constrained systems in curved spaces can be used to construct, new, more transparent descriptions of the gauge field theory.

In this paper, we combine a set of ideas available in the literature to construct the parent theory of higher spin gauge fields on AdS: the use of an embedding space with vielbeins and connections [26, 18, 27, 28] (see also [29] for a review) and a Fedosov-type approach for constrained systems [30 – 34] in order to achieve a generally covariant description. The resulting parent theory is completely natural from a geometrical point of view and admits a transparent algebraic structure with the simplest possible numerical factors. We first reduce to the BRST based "metric-like" description of higher spins on AdS [35 – 37] that is directly related to Fronsdal's original formulation by computing the cohomology of the space-time part of the parent differential. Instead of the subalgebra of $sp(4)$ that underlies the parent theory in the embedding space, the algebraic structure of the BRST formulation in intrinsic coordinates on AdS is that of an open algebra and much more involved.

We then analyze the reduction to Vasiliev's unfolded formulation [38, 26] by computing the cohomology of the algebraic part of the BRST charge and show how the numerical factors and projectors of the unfolded formulation of [38, 26] arise during this reduction.[1] Along the way, we construct various new intermediate descriptions with less variables but more complicated structure. Besides the absence of projectors and coefficients involving counting operators the advantage of the parent formulation and some of its intermediate reductions is that, at the price of introducing additional target space ghost variables, the theory is described by a single BRST operator. This is in contrast to the unfolded formulation [38, 26], where the higher spin theory is described by two differentials, one in the sector of higher spin connections, the other in the sector of curvatures, along with algebraic constraints relating these two sectors.

We hope that the parent theory or one of its intermediate reductions will be useful for resolving the above-mentioned problem of compatibility between Lagrangian and interaction. Because of the way the parent theory is obtained from the theory of higher spin gauge fields in flat space (in one dimension more and with 2 additional constraints), we also expect that problems such as the classification of all global symmetries or the AdS/CFT correspondence in this context might be more tractable. Finally, because the BRST formulation provides a unified framework for both higher spin gauge fields and string (field) theory, it might be worthwhile to explore to what extent successful techniques can be transfered from one subject to the other.

The paper is organized as follows: in the next section, we briefly recall how to associate a gauge field theory to a BRST first-quantized system. We also discuss some reduction techniques on the first quantized level and the relation with generalized auxiliary fields. Finally, we comment on the existence of Lagrangians associated with BRST field theories. In section 3, we give the details on the embedding and the covariantization procedure by constructing the parent theory for a scalar particle on AdS. We also discuss its reductions to standard and unfolded form. The inclusion of additional internal degrees of freedom to get our main result, the parent theory for higher spin gauge fields on AdS, is then straightforward and done in section 4. The explicit reductions are more involved. We summarize the main steps in the rest of section 4. Mathematical and technical details on reductions are relegated to the appendix.

JHEP08(2006)013

2. Gauge field theories associated to first-quantized systems

2.1 BRST differential and equations of motion

Let us briefly recall some basic facts about free field theories associated to BRST first-quantized systems with vanishing Hamiltonian. More detailed expositions can be found for instance in [39, 40] and in [25], which we follow here.

Suppose we are given with a quantum BRST system whose space of states is the space of sections $\Gamma(\boldsymbol{\mathcal{H}})$ of a vector bundle $\boldsymbol{\mathcal{H}}$ over a space-time manifold $\mathcal{X}$. Locally, the space

[1] Note that the unfolded formulation can also be directly reduced to Fronsdal's original formulation, see e.g. [29].

of states can be identified with functions on $\mathcal{X}$ taking values in the graded superspace $\mathcal{H}$. The degree is identified with the ghost number and denoted by $\mathrm{gh}(\cdot)$. The BRST operator $\Omega : \Gamma(\boldsymbol{\mathcal{H}}) \to \Gamma(\boldsymbol{\mathcal{H}})$, $\mathrm{gh}(\Omega) = 1$ is assumed to be a Grassmann odd differential operator of finite order. Locally, it is a differential operator with coefficients in linear operators on $\mathcal{H}$. In what follows such a BRST system is referred to as a BRST first-quantized system $(\Omega, \Gamma(\boldsymbol{\mathcal{H}}))$.

A local gauge field theory is associated to $(\Omega, \Gamma(\boldsymbol{\mathcal{H}}))$ in the following way. If e_A is a real frame of the bundle $\boldsymbol{\mathcal{H}}$, a generic section is given by $\boldsymbol{\phi} = \boldsymbol{\phi}^A(x)e_A$. One then introduces an independent field $\psi^A(x)$ for each component, with Grassmann parity and the ghost number prescribed by $|\psi^A| = |e_A|$ and $\mathrm{gh}(\psi^A) = -\mathrm{gh}(e_A)$. All component fields are combined into a single string field

$$\Psi = e_A \otimes \psi^A \,, \tag{2.1}$$

understood (locally) as an element of the tensor product of $\mathcal{H}$ and the algebra of local functions, i.e., functions in ψ^A and their space-time derivatives, see [25] for details. The field theory associated with the first-quantized system $(\Omega, \Gamma(\boldsymbol{\mathcal{H}}))$ is determined by the BRST differential $s\Psi = \Omega\Psi$ or, in terms of components, $s\psi^A = \Omega^A_B \psi^B$. Differential s is extended to arbitrary local functions by requiring that s satisfies the Leibniz rule and commutes with the total derivative

$$\partial_\mu = \frac{\partial}{\partial x^\mu} + \psi^A_{,\mu} \frac{\partial}{\partial \psi^A} + \psi^A_{,\mu\nu} \frac{\partial}{\partial \psi^A_{,\nu}} + \cdots \,. \tag{2.2}$$

In particular, the equations of motion have the form $s\Psi^{(-1)} = 0 \Leftrightarrow \Omega\Psi^{(0)} = 0$ while the gauge transformations are identified with $\delta\Psi^{(0)} = \Omega\Psi^{(1)}$ where ghost-number-one fields in $\Psi^{(1)}$ are replaced with gauge parameters. Here and in what follows we use the decomposition $\Psi = \sum_n \Psi^{(n)}$ of a string field into components $\Psi^{(n)}$ containing fields at ghost number n.

2.2 Reductions

Consider a not necessarily linear nor Lagrangian BRST gauge field theory described by a differential s, understood as a vector field on the space of fields ψ^A and their derivatives. The differential s is assumed to be local, i.e., $s\psi^A$ involve derivatives of finite order, and to be commuting with the total derivative ∂_μ. Even in this more general non linear context, it is still useful to combine all the fields into a string field Ψ. The equations of motion for the physical fields are then given by $s\Psi^{(-1)}|_{\Psi^k=0,\, k\neq 0} = 0$ while the gauge symmetries are determined by $\delta\Psi^{(0)} = s\Psi^{(0)}|_{\Psi^{(k)}=0,\, k\neq 0,1}$ with ghost-number-1 component fields of $\Psi^{(1)}$ replaced by gauge parameters.

Suppose that, after an invertible change of coordinates involving derivatives if necessary, the set of fields ψ^A splits into φ^α, w^a, v^a such that equations $sw^a|_{w^a=0} = 0$ (understood as algebraic equations in the space of fields and their derivatives) are equivalent to $v^a = V^a[\varphi^\alpha]$, i.e., can be algebraically solved for fields v^a. One then says that fields w, v are generalized auxiliary fields. The field theory described by s is then equivalent to that described by the reduced differential $\widetilde{s}$ acting on the space of fields φ^α and their derivatives and defined by $\widetilde{s}\varphi^\alpha = s\varphi^\alpha|_{w^a=0,\, v^a=V^a[\varphi]}$ (see [25] for more details). In the Lagrangian framework, fields w, v are in addition required to be second-class constraints in the antibracket sense. In this context, generalized auxiliary fields were originally proposed in [41]. Note that generalized auxiliary fields comprise both standard auxiliary fields and pure gauge degrees of freedom, together with associated ghost and antifields.

In the case where the gauge field theory is a linear theory associated with a BRST first-quantized system $(\Omega, \Gamma(\boldsymbol{\mathcal{H}}))$, one can proceed with the reductions at the first-quantized level. To identify a first-quantized counterpart of elimination of generalized auxiliary fields, we need to recall the notion of consistent reduction of a first-quantized gauge system discussed in [25]. In order to do so, we use the concept of algebraic invertibility: a differential operator $O : \Gamma(\boldsymbol{\mathcal{H}}) \to \Gamma(\boldsymbol{\mathcal{H}})$ is algebraically invertible iff it is invertible in the space of differential operators of (graded) finite order. In terms of a local frame a differential operator has the form $O = O^A_B(x, \frac{\partial}{\partial x})$ so that $O\boldsymbol{\phi} = O(e_A \boldsymbol{\phi}^A) = e_B O^B_A \boldsymbol{\phi}^A$ for $\boldsymbol{\phi} \in \Gamma(\boldsymbol{\mathcal{H}})$. Note that the derivative-independent part of an algebraically invertible operator is an invertible matrix.

Proposition 2.1. *Let $\boldsymbol{\mathcal{H}}$ decompose into a direct sum of vector bundles $\boldsymbol{\mathcal{H}} = \boldsymbol{\mathcal{E}} \oplus \boldsymbol{\mathcal{G}} \oplus \boldsymbol{\mathcal{F}}$ and the component $\overset{\boldsymbol{\mathcal{GF}}}{\Omega} = \mathcal{P}_{\boldsymbol{\mathcal{G}}} \Omega \mathcal{P}_{\boldsymbol{\mathcal{F}}}$, with $\mathcal{P}_{\boldsymbol{\mathcal{G}}}$,$\mathcal{P}_{\boldsymbol{\mathcal{F}}}$ denoting the projector to $\Gamma(\boldsymbol{\mathcal{G}})$, resp. $\Gamma(\boldsymbol{\mathcal{F}})$, be algebraically invertible as an operator from $\Gamma(\boldsymbol{\mathcal{F}})$ to $\Gamma(\boldsymbol{\mathcal{G}})$. Then the system $(\Omega, \Gamma(\boldsymbol{\mathcal{H}}))$ can be consistently reduced to $(\widetilde{\Omega}, \Gamma(\boldsymbol{\mathcal{E}}))$ with*

$$\widetilde{\Omega} = (\overset{\boldsymbol{\mathcal{EE}}}{\Omega} - \overset{\boldsymbol{\mathcal{EF}}}{\Omega} (\overset{\boldsymbol{\mathcal{GF}}}{\Omega})^{-1} \overset{\boldsymbol{\mathcal{GE}}}{\Omega}) \qquad \widetilde{\Omega} : \Gamma(\boldsymbol{\mathcal{E}}) \to \Gamma(\boldsymbol{\mathcal{E}}) \,. \tag{2.3}$$

In this case, the gauge field theories associated with $(\Omega, \Gamma(\boldsymbol{\mathcal{H}}))$ and $(\widetilde{\Omega}, \Gamma(\boldsymbol{\mathcal{E}}))$ are related by elimination of generalized auxiliary fields.

In appendix A we recall a useful proposition which allows one to systematically study various consistent reductions and discuss the relation with the so-called $\mathcal{D}$-module approach to linear partial differential equations.

To conclude this discussion of consistent reductions in first quantized terms, let us note that this procedure controls the problem of identifying generalized auxiliary fields in the non linear case as well. Indeed, suppose that the non linear theory corresponds to a consistent deformation of a linear theory associated to $(\Omega, \Gamma(\boldsymbol{\mathcal{H}}))$, i.e., the non linear BRST differential has the form

$$s = s_0 + g s_1 + g^2 s_2 + \cdots \,, \qquad s^2 = 0 \tag{2.4}$$

with $s_0 \Psi = \Omega \Psi$ the free BRST differential and g a coupling constant understood as formal deformation parameter. Now, if the fields of the theory split into w^a, v^a, φ^α so that $s_0 w^a = 0$ can be algebraically solved as $v^a = V^a_0[\varphi]$ at $w_a = 0$, i.e., if w, v are generalized auxiliary fields of the free theory, it is then easy to see that they are also generalized auxiliary fields for the deformed theory. Namely, at $w = 0$ equations $s w^a = 0$ can be algebraically solved as

$$v^a = V^a_0[\varphi] + g V^a_1[\varphi] + \cdots \,, \tag{2.5}$$

order by order in g. Note however that in this setting all quantities, such as the reduced BRST differential for instance, are formal power series in the deformation parameter g. In particular, even if s is polynomial, the reduced differential $\widetilde{s}$ can be an infinite series whose convergence is a separate question.

2.3 Lagrangians

Whenever there exists an inner product that makes the BRST operator Ω hermitian, the action that gives rise to the equations of motion is

$$\boldsymbol{S}^{\mathrm{ph}}[\Psi^{(0)}] = -\frac{1}{2}\langle \Psi^{(0)}, \Omega\Psi^{(0)}\rangle\,, \tag{2.6}$$

while the functional

$$\boldsymbol{S}[\Psi] = -\frac{1}{2}\langle \Psi, \Omega\Psi\rangle \tag{2.7}$$

is the Batalin-Vilkovisky master action [42 – 44, 13] associated with (2.6).

As in the beginning of the previous subsection, consider a not necessarily Lagrangian or linear BRST gauge field theory described by a BRST differential s and let us also assume that the set of fields ψ^A splits into fields φ^α, w^a, and v^a such that w^a and v^a are generalized auxiliary fields. Let $\widetilde{s}$ be the reduced BRST differential acting on the space of fields φ^α according to $\widetilde{s}\varphi^\alpha = s\varphi^\alpha|_{w^a=0,\, v^a=V^a[\varphi]}$.

Suppose now in addition that the reduced system described by $\widetilde{s}$ is Lagrangian which, on the level of the master action, is expressed through the existence of an antibracket $\big(\cdot,\cdot\big)_{\mathrm{red}}$ on the space of local functions in $\varphi^\alpha, \partial.\varphi_\alpha, \ldots$ such that $\widetilde{s}$ is generated by a master action $\widetilde{S}[\varphi]$, i.e., $\widetilde{s} = \big(\widetilde{S},\cdot\big)_{\mathrm{red}}$. Note that under appropriate regularity conditions, this is in fact equivalent to the existence of a standard Lagrangian for the equations $s\Psi^{(-1)}|_{\Psi^{(k)}=0,\, k\neq 0} = 0$. Under these assumptions, one can show that the original theory described by s can be also made Lagrangian by introducing "generalized" Lagrange multipliers. Generalized Lagrange multipliers are related to ordinary Lagrange multipliers in the same way as generalized auxiliary fields are related to ordinary ones: they are Lagrange multiplies on the level of the master action, instead of the classical action.

In order to see this, let us introduce adapted coordinates $\varphi^\alpha, w^a, v^a = sw^a$ as new independent coordinates on the space of fields. Moreover one can always redefine φ^α such that $s\varphi^\alpha$ are functions only of φ^α and their derivatives (see [25] for a proof). In the new coordinate system the differential s takes the form

$$s = s^\alpha[\varphi]\frac{\partial}{\partial\varphi^\alpha} + v^a\frac{\partial}{\partial w^a} + \cdots\,, \tag{2.8}$$

where dots denote the terms acting on derivatives.

The generalized Lagrange multipliers are then the new fields v^*_a and w^*_a with $|v^*_a| = |v^a| + 1$, $|w^*_a| = |w^a| + 1$ and $\mathrm{gh}(v^*_a) = -\mathrm{gh}(v^a) - 1$, $\mathrm{gh}(w^*_a) = -\mathrm{gh}(w^*_a) - 1$. The extended space is equipped with the following antibracket structure

$$\big(\varphi^\alpha, \varphi_\beta\big) = \big(\varphi^\alpha, \varphi_\beta\big)_{\mathrm{red}}\,, \qquad \big(v^a, v^*_b\big) = \delta^a_b\,, \qquad \big(w^a, w^*_b\big) = \delta^a_b\,, \tag{2.9}$$

with all the other basic antibrackets vanishing. The bracket is extended in the standard way (see e.g. [45]) to general local functions such that it satisfies the Leibnitz rule for the second argument and commutes with the total derivative acting on the second argument. The master action that describes the Lagrangian structure of the original theory is then given by

$$S = \widetilde{S} - \int d^d x \, v^a w^*_a \,. \tag{2.10}$$

It obviously satisfies the master equations $\frac{1}{2}\big(S, S\big) = 0$. If f does not depend on v^*, w^* and their derivatives then

$$\big(S, f\big) = sf \,, \tag{2.11}$$

where s is the original BRST differential.

Furthermore, fields v, w, v^*, w^* are obviously generalized auxiliary fields in the sense of [41]. Indeed, the equations of motion obtained by varying with respect to fields v and w^* can be algebraically solved for these variables. The reduced master action is $\widetilde{S}[\varphi]$ which establishes the equivalence of the extended and the reduced theories as Lagrangian field theories.

3. Scalar particle on AdS

3.1 (A)dS space as an embedding

We take the standard approach and describe gauge systems on constant curvature spaces by embedding the latter in a flat pseudo-Euclidean space. More precisely, we consider the surface $\mathcal{X}_0 \subset \mathbb{R}^{d+1}$ described by

$$\eta_{AB} X^A X^B + l^2 = 0 \,, \tag{3.1}$$

where X^A, $A = 0, \ldots, d$ stand for the standard coordinates in $\mathbb{R}^{d+1}$ while the metric is chosen as $\eta_{AB} = diag(-1, 1, \ldots, 1, -1)$. When $l^2 > 0$, the manifold $\mathcal{X}_0$ describes AdS space, the case $l^2 < 0$ corresponds to dS space. In what follows we explicitly consider the case where $\mathcal{X}_0$ is AdS space, but the analysis remains the same for other constant curvature spaces.

The main advantage of an embedding space over an intrinsic description is the transparent form of the isometries. Similarly, in the higher spin gauge field context, the characterization of the "vacuum symmetries", i.e., of the gauge transformations that leave the background solution invariant, is considerably simplified when one uses an embedding space (see e.g. [29]).

3.2 BRST operator

To demonstrate the approach in the most simple case, we consider the quantum theory of a massless scalar particle. At the classical level the phase space is just given by flat space with coordinates X^A, P_A subjected to the standard Poisson bracket relations. The effective phase space of a particle on $\mathcal{X}_0$ is described by the second class constraints

$$X^2 + l^2 = 0 \,, \qquad XP = 0 \tag{3.2}$$

together with the mass-shell constraint $P^2 = 0$. In order to have a description in terms of first class constraints only, the geometrical constraint $X^2 + l^2 = 0$ is excluded from the initial set of constraints and treated as a partial gauge fixing condition (see e.g. [46]). Note that in terms of wave functions this reproduces the well known approach of [4]. As a result, one has the following set of first class constraints

$$L = P_A \eta^{AB} P_B \,, \qquad M = P_A X^A \,, \tag{3.3}$$

which form a closed algebra $\{L, M\} = -2L$.

In principle, one can construct the quantum theory by treating P, X as operators represented on functions in X and build the associated gauge field theory, which then describes a scalar field on $\mathcal{X}_0$. Indeed, in this representation the constraint XP completely fixes the radial dependence of wave functions which then can be considered as fields on $\mathcal{X}_0$. However, we now take a different route and first extend the constrained system even further. What we want is an explicitly covariant formulation of the system, in terms of bundles on $\mathcal{X}_0$ with fibers related to the embedding space. For this purpose, we generalize the parent theory of [25] to the case of constant curvature spaces.

The extension amounts to introducing new variables Y^A, momenta $\bar{P}_B$ and postulating the following Poisson bracket relations on the extended space:

$$\{\bar{P}_A, X^B\} = -\delta^B_A \,, \qquad \{P_A, Y^B\} = -\delta^B_A \,. \tag{3.4}$$

The original phase space can then be identified with the constrained surface determined by the following second class constraints:

$$P_A - \bar{P}_A = 0 \,, \qquad Y^A = 0 \,. \tag{3.5}$$

Indeed, computing the Dirac bracket and solving constraints one arrives at the original phase space. Taking into account the original constraints L, M one finds that the equivalent set of constraints is given by

$$P_A - \bar{P}_A = 0 \,, \qquad Y^A = 0 \,, \qquad (X^A + Y^A) P_A = 0 \,, \qquad P^A P_A = 0 \,. \tag{3.6}$$

One then observes that all constraints with $Y^A = 0$ excluded are first class.

Passing to the quantum description one treats the variables $P, \bar{P}, X, Y$ as quantum operators with the following commutation relations[2]

$$[\bar{P}_A, X^B] = -\delta^B_A \,, \qquad [P_A, Y^B] = -\delta^B_A \,, \tag{3.7}$$

and introduces the Grassmann odd ghost variables Θ^A, μ, c_0 with $\mathrm{gh}(\Theta^A) = \mathrm{gh}(\mu) = \mathrm{gh}(c_0) = 1$ and their conjugate momenta:

$$[b_0, c_0] = -1 \,, \qquad [\rho, \mu] = -1 \,, \qquad [\mathcal{P}_A, \Theta^B] = -\delta^B_A \,. \tag{3.8}$$

[2]For later convenience, we deviate from the standard convention, used for instance in [25], and choose $\text{momenta}^{\text{here}} = \imath\, \text{momenta}^{\text{stand}}$. We will also use $\Omega^{\text{here}} = \imath \Omega^{\text{stand}}$ below.

Finally, the nilpotent BRST operator that takes into account the first-class subset of (3.6) reads as

$$\Omega = \Theta^A (P_A - \bar{P}_A) + c_0 P^2 + \mu (X^A + Y^A) P_A - 2 c_0 \mu b_0 \,. \tag{3.9}$$

It can be worth mentioning that the easiest way to arrive at (3.9) is to start from the BRST operator

$$\Omega_{\text{stand}} = \Theta^A \bar{P}_A + c_0 P^2 + \mu X^A P_A - 2 c_0 \mu b_0 \,, \tag{3.10}$$

which decomposes into independent pieces: the BRST operator $\Theta^A \bar{P}_A$ eliminating pure gauge variables $\bar{P}, Y, \Theta, \mathcal{P}$ and the standard BRST operator for constraints L, M. At this stage, the commutation relations are standard: $[Y^A, \bar{P}_B] = \delta^A_B$ and $[X^A, P_B] = \delta^A_B$. The BRST operator (3.9) with the commutation relations (3.7) are then obtained by performing the change of variables $X \to X+Y$ and $\bar{P}_A \to P_A - \bar{P}_A$. The advantage of the more involved arguments leading to (3.9) is that they can be naturally generalized to the case of a curved phase space and to the case where the allowed functional spaces for X and Y variables are different. In particular they remain valid if one allows for formal power series in Y variables in observables and wave functions.

Before reducing to the surface by imposing the geometrical constraint $X^2 + l^2 = 0$, it is convenient to recast the system in a more geometrical way. This is achieved by introducing arbitrary coordinates $\underline{X}^{\underline{A}}$ through $X^A = X^A(\underline{X}^{\underline{B}})$. An $\underline{X}^{\underline{B}}$ dependent rotation in the fiber, $Y' = \Lambda Y$, $P' = P \Lambda^{-1}$ can then be completed to a canonical transformation if $\bar{P}_{\underline{A}} = \bar{P}'_{\underline{A}} - W^B_{\underline{A}C} Y'^C P'_B$. After dropping the primes, the transformed BRST operator becomes

$$\Omega = \Theta^{\underline{A}} (E^B_{\underline{A}} P_B - \bar{P}_{\underline{A}}) + \Theta^{\underline{A}} W^C_{\underline{A}B} Y^B P_C + c_0 P^2 + \mu (V^A + Y^A) P_A - 2 c_0 \mu b_0 \,, \tag{3.11}$$

where

$$W = -(d\Lambda)\Lambda^{-1} \,, \qquad V = \Lambda X \,, \qquad E = \Lambda dX \,. \tag{3.12}$$

More generally, instead of $\mathbf{R}^{d+1}$ one can consider a $d+1$ flat pseudo-Riemannian manifold $\mathcal{X}$ with coordinates $\underline{X}^{\underline{A}}$ and introduce $\boldsymbol{\mathcal{V}}(\mathcal{X})$, the vector bundle associated with the orthonormal frame bundle and isomorphic to $T(\mathcal{X})$. Fiber coordinates on $\boldsymbol{\mathcal{V}}(\mathcal{X})$ are denoted by Y^A and the flat fiber metric is $\eta = diag(-1, 1, \ldots, 1, -1)$. The fiber-wise isomorphism (vielbein) between $\boldsymbol{\mathcal{V}}(\mathcal{X})$ and $T\mathcal{X}$ is denoted by E. One further extends the phase space to the cotangent bundle $T^*(\boldsymbol{\mathcal{V}}(\mathcal{X}))$, with variables $\bar{P}_{\underline{A}}, P_A$ being coordinates on the fibers. Finally, ghost variables $\Theta^{\underline{A}}, \mathcal{P}_{\underline{A}}$ are introduced extending the phase space to $T^*(\boldsymbol{\mathcal{V}}(\mathcal{X}) \oplus \Pi T\mathcal{X})$, with Π denoting parity reversing.

At the quantum level the operators satisfy the commutation relations

$$[\bar{P}_{\underline{A}}, \underline{X}^{\underline{B}}] = -\delta^{\underline{B}}_{\underline{A}} \,, \qquad [P_A, Y^B] = -\delta^B_A \,, \qquad [\mathcal{P}_{\underline{A}}, \Theta^{\underline{B}}] = -\delta^B_A \,, \tag{3.13}$$

originating from the canonical Poisson brackets on $T^*(\boldsymbol{\mathcal{V}}(\mathcal{X}) \oplus \Pi T\mathcal{X})$. The BRST operator (3.11) is nilpotent and (at least locally) describes a particle on a submanifold $\mathcal{X}_0 \subset \mathcal{X}$ transversal to the vector field $V^A E^{\underline{B}}_A \frac{\partial}{\partial \underline{X}^{\underline{B}}}$ provided (i) the connection W on $\boldsymbol{\mathcal{V}}(\mathcal{X})$ is flat and compatible with the fiber-wise metric and (ii) the vielbein is nondegenerate, covariantly constant, and given by the covariant derivative of a fixed section V of $\boldsymbol{\mathcal{V}}(\mathcal{X})$:

$$\begin{aligned} W\eta + \eta W^T = 0\,, \qquad & dW + W^2 = 0\,, \\ dE + WE = 0\,, \qquad & E = dV + WV. \end{aligned} \tag{3.14}$$

Let us stress that in this formulation, the BRST operator (3.11) is defined invariantly. Indeed, by making use of the transformation properties of the components of W, V, E and $Y, P, \bar{P}, \Theta, \mathcal{P}$ under changes of local coordinates on $\mathcal{X}$ and local frames of $\boldsymbol{\mathcal{V}}(\mathcal{X})$, Ω is easily seen to be invariant.

The extension just described is a very simple example of so-called Fedosov quantization [30] extended to the case of constrained systems. More precisely, it corresponds to a version of Fedosov quantization adapted to the case of cotangent bundles which was first considered in [31]. The extension to the case of systems with constraints and the interpretation in terms of BRST theory were developed in [32 − 34]. From this perspective a natural generalization is to allow the connection W to be non-flat which can be appropriate when considering curved phase space. In the case at hand taking W flat is the simplest and natural option. Curvature will now be introduced by restricting to a submanifold.

3.3 Reduction to the surface

The essential feature of the extended system described by (3.9) is that the space-time coordinates X^A and their associated momenta are pure gauge degrees of freedom. It should be noted, however, that the elimination of these degrees of freedom is in general valid only locally and that geometrical data is lost in the process. Moreover, at the level of associated field theories such an elimination leads to theories which are not equivalent as local field theories, as it relates for instance theories that live in different space-time dimensions. This is not so for the coordinate r transversal to the surface $X^2+l^2=0$, which is not considered as a proper space-time coordinate from the very beginning because by construction the system effectively lives on this surface. In this sense r can be consistently eliminated both at the first-quantized level and at the level of the associated field theory.

More technically, we first take a coordinate system $\underline{X}^{\underline{A}} = (x^\mu, r)$ adapted to the surface: $r = \sqrt{-X^2}$ and $X^A \frac{\partial}{\partial X^A} x^\mu = 0$. In the new coordinate system, the BRST operator takes the form:

$$\begin{aligned} \Omega = \theta^{(r)}(E^A_{(r)} P_A - \bar{p}_{(r)}) + \theta^\mu(E^A_\mu P_A - \bar{p}_\mu) + \theta^{(r)} W^A_{(r)\,B} Y^B P_A + \\ + \theta^\mu W^A_{\mu B} Y^B P_A + c_0 P^2 + \mu(V^A P_A + Y^A P_A) - 2c_0\mu b_0\,, \end{aligned} \tag{3.15}$$

with E, W, V defined as in (3.12), $V^2 = -r^2$, and superscript (r) denoting the component along $\frac{\partial}{\partial r}$ of a tangent vector.

Suppose the system to be quantized in the coordinate representation for the variables $r, \theta^{(r)}, \bar{p}_{(r)}, \mathcal{P}_{(r)}$. In a neighborhood of $r = l$, the variables $\theta^{(r)}$ and $r - l$ form contractible pairs, or in other words, condition $r = l$ can be considered as a gauge fixing condition. It follows that the system can be reduced by solving the linear second class constraints

$r = l, \theta^{(r)} = 0, \mathcal{P}_{(r)} = 0, \bar{p}_{(r)} = 0$ in the BRST operators and putting $\theta^{(r)} = 0$ and $r = l$ in the wave functions. The reduced BRST operator is given by

$$\Omega^{\mathrm{T}} = \theta^\mu (e_\mu^A P_A - \bar{p}_\mu) + \theta^\mu \omega_{\mu B}^A Y^B P_A + c_0 P^2 + \mu (V^A + Y^A) P_A - 2c_0 \mu b_0 \,, \tag{3.16}$$

where now $\omega_{\mu B}^A = W_{\mu B}^A(x, l)$, $e_\mu^A = E_\mu^A(x, l)$, $V^A = V^A(x, l)$ and satisfy

$$\begin{aligned} d\omega_A^B + \omega_C^B \omega_A^C &= 0 \,, & de^A + \omega_B^A e^B &= 0 \,, \\ dV^A + \omega_B^A V^B &= e^A , & V^A V_A + l^2 &= 0 \,. \end{aligned} \tag{3.17}$$

These define the compatible flat connection, the vielbein, and the fixed section of a vector bundle $\boldsymbol{\mathcal{V}}(\mathcal{X}_0)$ that can be identified with the bundle $\boldsymbol{\mathcal{V}}(\mathcal{X})$ pulled back to the submanifold $\mathcal{X}_0 \subset \mathcal{X}$. That such a flat connection ω_C^B, covariantly constant vielbein e^A and "compensator" V^A are most useful to describe tensor fields on AdS has been originally understood in [27, 28, 26, 18].

In terms of the associated field theory, the fields associated to all r, θ^r dependent states are then generalized auxiliary fields in the sense of [25], provided one considers r as an internal degree of freedom rather than a space-time coordinate. A more detailed proof of this fact can be found in [47] in the nonlinear case.[3] Note that, contrary to the other reductions considered in this paper, the one given here merely serves to define and motivate the parent theory on AdS and does not mean that the parent theory on AdS is equivalent, as a local field theory, to the flat theory in one dimension more.

The representation space $\Gamma(\boldsymbol{\mathcal{H}}^{\mathrm{T}})$ for the quantum system is chosen to be the space of "functions" in x, Y, c_0, μ, θ which are formal power series in Y with coefficients in smooth functions in x and polynomials in the ghosts. In terms of the representation space the BRST operator acts as follows:

$$\Omega^{\mathrm{T}} = \boldsymbol{d} - \theta^\mu \omega_{\mu B}^A Y^B \frac{\partial}{\partial Y^A} - \theta^\mu e_\mu^A \frac{\partial}{\partial Y^A} + c_0 \frac{\partial^2}{\partial Y^A \partial Y_A} - \mu (V^A + Y^A) \frac{\partial}{\partial Y^A} - 2\mu c_0 \frac{\partial}{\partial c_0}, \tag{3.18}$$

where $\boldsymbol{d} = \theta^\mu \frac{\partial}{\partial x^\mu}$ can be considered as the De Rham differential provided one identifies θ^μ and dx^μ. With this choice, the quantum system described by the BRST operator (3.16) is a parent system for a scalar particle on AdS. Indeed, we will now show that it can be reduced both to the standard and the unfolded description of a particle on AdS.

To proceed with the reductions, we first note that the BRST operator (3.16) and hence (3.18) also do not depend on the choice of local coordinates on $\mathcal{X}_0$ and local frames of $\boldsymbol{\mathcal{V}}(\mathcal{X}_0)$. A particulary useful choice is to take the orthonormal local frame such that $V^A = l\delta_{(d)}^A$ for which $e^A = l\omega_{(d)}^A$ implying $e^{(d)} = 0$. Here and in what follows $Z^{(d)}$ denotes the $d+1$-th component of a section Z, e.g. $Z^A Z_A = Z^a Z_a + Z^{(d)} Z_{(d)} = Z^a Z_a - Z^{(d)} Z^{(d)}$. With this choice,

$$\omega_B^A = \begin{pmatrix} \omega_b^a & \frac{1}{l} e^a \\ \frac{1}{l} e_b & 0 \end{pmatrix} ,$$

[3] The parent formulation just described can be understood as a generalization of the unfolded formulation [20, 21, 48] in which the auxiliary role of space-time coordinates has been understood in [22].

and e^a, $a = 0, \ldots, d-1$, is the standard AdS vielbein, with associated connection $\omega^a{}_b$:

$$ds^2_{\mathrm{AdS}} = \eta_{ab}\, e^a \otimes e^b, \qquad de^a + \omega^a{}_b e^b = 0,$$
$$R^{ab} \equiv d\omega^{ab} + \omega^a{}_c \omega^{cb} = -\frac{1}{l^2} e^a e^b\,, \tag{3.19}$$

and the BRST operator becomes

$$\Omega^{\mathrm{T}} = \theta^\mu (e^a_\mu P_a - \bar{p}_\mu) + \theta^\mu \omega^A_{\mu\, B} Y^B P_A + c_0 P^2 + \mu(l P_{(d)} + Y^A P_A) - 2c_0 \mu b_0. \tag{3.20}$$

3.4 Reduction to standard description

It is straightforward to show that the system $(\Omega^{\mathrm{T}}, \Gamma(\boldsymbol{\mathcal{H}}^{\mathrm{T}}))$ can be consistently reduced to the standard description of a particle on AdS or, more precisely, to the system $(\widetilde{\Omega}, \Gamma(\boldsymbol{\mathcal{E}}))$ where $\Gamma(\boldsymbol{\mathcal{E}})$ is the space of x, c_0 dependent functions and

$$\widetilde{\Omega}\boldsymbol{\phi}_0 = c_0 \Box_{\mathrm{AdS}} \boldsymbol{\phi}_0, \qquad \Box_{\mathrm{AdS}} \boldsymbol{\phi}_0 = \eta^{\mu\nu} (\partial_\mu \partial_\nu - \Gamma^\rho_{\mu\nu} \partial_\rho) \boldsymbol{\phi}_0 \tag{3.21}$$

for $\boldsymbol{\phi}_0 \in \Gamma(\boldsymbol{\mathcal{E}})$ and where $\Gamma^\rho_{\mu\nu} = e^\rho_a \omega^a_{\mu b} e^b_\nu + e^\rho_a \partial_\mu e^a_\nu$. Details can be found in appendix B.1.

3.5 Reduction to unfolded form

According to [25], the parent system can be reduced to its unfolded form by computing the cohomology of the part of the BRST differential that does not involve space-time ghosts θ^μ. One takes as degree minus the target space ghost number, i.e., minus the degree in c_0, μ, according to which the BRST operator decomposes as $\Omega = \Omega_{-1} + \Omega_0$, with

$$\Omega_{-1} = c_0 \Box + \mu h - 2\mu c_0 \frac{\partial}{\partial c_0}, \qquad \Omega_0 = \boldsymbol{d} - \omega^A_B Y^B \frac{\partial}{\partial Y^A} + \sigma\,, \tag{3.22}$$

where

$$h = -(Y^A + V^A) \frac{\partial}{\partial Y^A}\,, \qquad \Box = \frac{\partial}{\partial Y^A} \frac{\partial}{\partial Y_A}\,, \qquad \sigma = -\theta^\mu e^A_\mu \frac{\partial}{\partial Y^A}\,. \tag{3.23}$$

For later convenience, let us set $Y^{(d)} = lz$ and $y^a = Y^a$. In appendix B.2, we will prove:

Proposition 3.1. *The cohomology of Ω_{-1} in the spacce $\mathcal{H}^{\mathrm{T}}$ of formal power series in Y^A with coefficients in polynomials in c_0, μ, θ^ν is given by*

$$H^0(\Omega_{-1}, \mathcal{H}^{\mathrm{T}}) \cong \mathcal{E} \subset \ker \Omega_{-1}\,, \qquad H^n(\Omega_{-1}, \mathcal{H}^{\mathrm{T}}) = 0 \quad n > 0\,, \tag{3.24}$$

where $\mathcal{E} = \ker \Box \cap \ker h$. In the frame where $V^A = l\delta^A_{(d)}$, $\mathcal{E}$ is canonically isomorphic to the space $\bar{\mathcal{E}}$ of traceless μ, c_0, z-independent elements. The isomorphism $\mathcal{K} : \bar{\mathcal{E}} \to \mathcal{E}$ is given by $\mathcal{K}^{-1}\phi = \mathcal{P}(\phi|_{z=0})$ for any $\phi \in \mathcal{E}$ where $\mathcal{P}$ denotes the projector to the traceless component (in the space of z-independent elements if $\phi = \phi_0 + (y^a y_a)\phi_1$ and $\Box\phi_0 = 0$ then $\mathcal{P}\phi = \phi_0$).

As recalled in appendix A, the reduced BRST operators is Ω_0 understood as acting in $\Gamma(\boldsymbol{\mathcal{E}})$ because the cohomology of Ω_{-1} is concentrated in one degree. For $\boldsymbol{\phi} \in \Gamma(\boldsymbol{\mathcal{E}})$ one then gets

$$\widetilde{\Omega}\boldsymbol{\phi} = (\nabla + \sigma)\boldsymbol{\phi}\,, \qquad \nabla = \boldsymbol{d} - \omega^A_B Y^B \frac{\partial}{\partial Y^A}\,. \tag{3.25}$$

The associated equations of motion take the form $(\nabla+\sigma)\Psi^{(0)}=0$ where $\Psi^{(0)}$ contains the fields associated with the θ^μ-independent states in $\mathcal{E}$, i.e., $\Psi^{(0)}=\Psi^{(0)}(x,Y)$ and $\Box\Psi^{(0)}=h\Psi^{(0)}=0$.

It is natural to rewrite the system in terms of $\bar{\mathcal{E}}$ valued fields. More precisely, in the frame where $V^A=l\delta^A_{(d)}$ the operators h and Ω_0 take the form

$$h=-Y^a\frac{\partial}{\partial Y^a}-(z+1)\frac{\partial}{\partial z}\,,\qquad \Omega_0=\boldsymbol{d}-\omega^a_{\ b}Y^b\frac{\partial}{\partial Y^a}+(z+1)\sigma-\frac{1}{l^2}e^a y_a\frac{\partial}{\partial z}\,. \tag{3.26}$$

For any $\phi_0\in\bar{\mathcal{E}}$, element $\phi=\mathcal{K}\phi_0$ reads as (see appendix B.2)

$$\phi=\frac{1}{(1+z)^n}\left(\phi_0+(y^a y_a)\frac{n(n+1)}{2l^2(d+2n)}\phi_0+\cdots\right)\,, \tag{3.27}$$

where $n=y^a\frac{\partial}{\partial y^a}$, the ratio is understood as a formal power series, and $\ldots$ denote terms of the form $(y^a y_a)^k\phi_0,\ k\geqslant 2$.

In terms of $\bar{\mathcal{E}}$-valued sections, the reduced BRST operator is given by $\Omega^{\mathrm{unf}}=\mathcal{K}^{-1}\widetilde{\Omega}\mathcal{K}$. Using (3.26) and (3.27), one finds as explicit expression

$$\Omega^{\mathrm{unf}}\boldsymbol{\phi}_0=\left[\boldsymbol{d}-\omega^a_{\ b}y^b\frac{\partial}{\partial y^a}+\sigma\right]\boldsymbol{\phi}_0-\frac{(n-1)(n+d-2)}{l^2(d+2n-2)}\mathcal{P}[e^a y_a\boldsymbol{\phi}_0]\,, \tag{3.28}$$

where $\boldsymbol{\phi}_0\in\Gamma(\bar{\boldsymbol{\mathcal{E}}})$ and $\mathcal{P}$ denotes the projector to the subspace of traceless elements. This expression coincides with the differential determining the unfolded form of the Klein-Gordon equation on AdS space proposed in [49].

4. Free higher spin gauge fields on AdS

4.1 The first-quantized model

Instead of the standard string inspired first-quantized description of higher spin gauge fields in flat [14–16] and in AdS space in intrinsic coordinates [36, 37], we follow here the strategy of the preceding section and construct a parent theory for higher spin gauge fields on constant curvature spaces by using an embedding. Namely, we incorporate the constraints describing the reduction to the surface into the flat first-quantized BRST system [14–16] in the embedding space $\mathbb{R}^{d+1}$.

With respect to the particle, the additional variables besides X^A, P_B are $a^A, a^{\dagger B}$, where $A=0,\ldots,d$. At the quantum level, these variables satisfy the commutation relations

$$[P_B,X^A]=-\delta^A_B,\qquad [a^A,a^{\dagger B}]=\eta^{AB}. \tag{4.1}$$

The constraints of the system are

$$\begin{aligned}\mathcal{L}&\equiv\eta^{AB}P_AP_B=0, & T&\equiv\eta_{AB}a^Aa^B=0,\\ \mathcal{S}^\dagger&\equiv-P_Aa^{\dagger A}=0, & \mathcal{S}&\equiv-P_Aa^A=0.\end{aligned} \tag{4.2}$$

For the ghost pairs (c_0, b_0), $(c^\dagger, b)$, $(c, b^\dagger)$, and ξ, π corresponding to each of these constraints, we take the canonical commutation relations in the form[4]

$$[b_0, c_0] = -1, \qquad [c, b^\dagger] = 1, \qquad [b, c^\dagger] = -1, \qquad [\pi, \xi] = -1\,. \tag{4.3}$$

The ghost-number assignments are

$$\begin{gathered}\mathrm{gh}(c_0) = \mathrm{gh}(c) = \mathrm{gh}(c^\dagger) = \mathrm{gh}(\xi) = 1,\\ \mathrm{gh}(b_0) = \mathrm{gh}(b) = \mathrm{gh}(b^\dagger) = \mathrm{gh}(\pi) = -1.\end{gathered} \tag{4.4}$$

The BRST operator is then given by

$$\Omega_0 = c_0\mathcal{L} + c^\dagger\mathcal{S} + \mathcal{S}^\dagger c + \xi T + c^\dagger c b_0 + 2\xi c b, \tag{4.5}$$

while the representation space consists of functions in X^A (on which P_A acts as $-\frac{\partial}{\partial X^A}$) with values in the "internal space" $\mathcal{H}_0$. The latter is the tensor product of the space $\mathcal{H}_{c_0,\xi}$ of functions in c_0, ξ (coordinate representation for (c_0, b_0) and (ξ, π)) and the Fock space for $(a^\dagger_A, a^A)$, $(c^\dagger, b)$, and $(c, b^\dagger)$ defined by

$$a^A|0\rangle = b|0\rangle = c|0\rangle = 0. \tag{4.6}$$

To reduce the system to AdS space, in addition to constraints (4.2), one needs to impose the "geometrical" constraints

$$X^2 + l^2 = 0\,, \quad X^A P_A = 0\,, \quad X^A a_A = 0\,, \quad X^A a^\dagger_A = 0\,, \tag{4.7}$$

which are second class.

Again, after first passing to an equivalent set of second class constraints, we will keep only half of them so that the remaining constraints together with (4.2) form a first class set, the other constraints being considered as partial gauge fixing conditions. More precisely, one first considers the constraints

$$X^A P_A + a^\dagger_A a^A = 0\,, \qquad X^A a_A = 0 \tag{4.8}$$

and checks that together with constraints (4.2), they form a closed algebra. Introducing new pairs of ghost variables μ, ρ and ν, τ with

$$\mathrm{gh}(\mu) = \mathrm{gh}(\nu) = 1\,, \qquad \mathrm{gh}(\rho) = \mathrm{gh}(\tau) = -1 \tag{4.9}$$

and commutation relations

$$[\rho, \mu] = -1\,, \quad [\tau, \nu] = -1\,, \tag{4.10}$$

one can incorporate all the constraints into a standard BRST operator. The resulting BRST system [35, 46] describes Fronsdal's higher spin gauge fields on AdS.

In order to construct the parent theory we introduce, as in the previous section, new variables Y^A, momenta $\bar{P}_A$ and ghost pairs $\Theta^A, \mathcal{P}_A$ with commutation relations:

$$[\bar{P}_A, X^B] = -\delta^B_A\,, \qquad [P_A, Y^B] = -\delta^B_A\,, \qquad [\mathcal{P}_A, \Theta^B] = -\delta^B_A\,. \tag{4.11}$$

[4]We use the "super" convention that $(ab)^\dagger = (-1)^{|a||b|} b^\dagger a^\dagger$.

The extended system is now described by the constraints

$$P_A - \bar{P}_A = 0\,, \tag{4.12}$$

and all original constraints understood as functions of P and extended by Y-dependent terms so as to commute with constraints (4.12). In fact only constraints $XP + a^\dagger a = 0$ and $XA = 0$ get corrected to $XP + XY + a^\dagger a = 0$ and $Xa + Ya = 0$ respectively. Finally, one constructs the following nilpotent BRST operator:

$$\begin{aligned}\Omega = \theta^A(P_A - \bar{P}_A) + c_0 P^2 - a^\dagger P c - c^\dagger P a + \xi a^2 + \\ + \mu[(X+Y)P + a^\dagger a] + \nu (X+Y) a + \text{terms cubic in ghosts}\,.\end{aligned} \tag{4.13}$$

4.2 Reduction to the surface and algebraic structure

Proceeding exactly in the way as in the case of the scalar particle, the BRST operator for the system pulled back to $\mathcal{X}_0$ is

$$\begin{aligned}\Omega^{\mathrm{T}} = \theta^\mu (e_\mu^B P_B - \bar{p}_\mu) + \theta^\mu \omega_{\mu}{}^{B}{}_{C}(Y^C P_B - a^{\dagger C} a_B) + \\ + c_0 \Box + S^\dagger c + c^\dagger S + \xi T + \mu h + \nu \bar{S}^\dagger + \text{terms cubic in ghosts}\,,\end{aligned} \tag{4.14}$$

where the following notations have been introduced for constraints

$$\begin{gathered} S = -aP\,, \qquad S^\dagger = -a^\dagger P\,, \qquad T = a^2\,, \qquad \Box = P^2\,, \\ \bar{S}^\dagger = (Y+V)a\,, \qquad h = a^\dagger a + (Y+V)P\,, \end{gathered} \tag{4.15}$$

and e^A, V^A, ω_A^B satisfy (3.17) and are considered components of the vielbein, the fixed section, and the connection in $\boldsymbol{\mathcal{V}}(\mathcal{X}_0)$. Note that variables $a^{\dagger A}$, a^A are, along with Y-variables, to be considered as coordinates on $\boldsymbol{\mathcal{V}}(\mathcal{X}_0)$.

The constraint algebra that determines the form of the terms cubic in ghosts reads as:

$$\begin{gathered} [S, S^\dagger] = \Box\,, \quad [h, \Box] = 2\Box\,, \quad [h, S^\dagger] = 2S^\dagger\,, \quad [S, \bar{S}^\dagger] = T\,, \quad [S^\dagger, \bar{S}^\dagger] = h\,, \\ [h, T] = -2T\,, \quad [h, \bar{S}^\dagger] = -2\bar{S}^\dagger\,, \quad [T, S^\dagger] = 2S\,, \qquad [\Box, \bar{S}^\dagger] = 2S\,, \end{gathered} \tag{4.16}$$

with all other commutators vanishing. It is not difficult to see that it is a subalgebra of $sp(4)$ identified in [25] as the algebraic structure underlying the parent theory of Fronsdal higher spin gauge fields in the flat space. Note however that in the flat case the subalgebra of $sp(4)$ entering the BRST operator does not contain the generators $\bar{S}^\dagger, h$ and all the $sp(4)$ generators are represented on variables $Y^\mu, a^{\dagger\mu}$ associated with d-dimensional tangent space. In the case at hand the variables $Y^A, a^{\dagger A}$ are associated with the $d+1$ dimensional embedding space.

Another important difference with the flat case is the shift $Y \to Y+V$ in the generators. Moreover, it follows from $dV^A + \omega^A_B V^B = e^A$ that in terms of $Y' = Y + V$ the expression for the BRST operator takes the form

$$\Omega^{\mathrm{T}} = -\theta^\mu \bar{p}_\mu + \theta^\mu \omega_{\mu}{}^{B}{}_{C}({Y'}^C P_B - a^{\dagger C} a_B) +$$

$$+ c_0\Box + S^\dagger c + c^\dagger S + \xi T + \mu h + \nu \bar{S}^\dagger + \text{terms cubic in ghosts}\,. \quad (4.17)$$

Because the BRST operator is polynomial in Y the change of coordinates $Y' = Y + V$ is legitimate and gives, in particular, the easiest way to check nilpotency explicitly. Indeed, in these terms, nilpotency immediately follows from the fact that W is a flat connection compatible with the metric η_{AB} and all generators (4.15) are build from $a^\dagger, a$ and Y', P with the indexes contracted with η_{AB}. Note that in general one is not allowed to do such a change of variables in the representation space where Y, P are represented on formal power series in Y.

In order to obtain a representation of all of $sp(4)$, one needs to add the following generators:

$$\bar{T} = -\frac{1}{4}(a^\dagger)^2\,, \quad \bar{S} = (Y+V)a^\dagger\,, \quad \bar{\Box} = (V+Y)^2\,, \quad h' = -a^\dagger a - \frac{d+1}{2}\,, \qquad (4.18)$$

which makes the total number of generators ten as it should be. Note that all the generators are invariant under the choice of a basis in the $d+1$ dimensional linear space provided the components of $Y, P, a^\dagger, a$ transform as (co)vectors and η_{AB} as components of a bilinear form. In particular, $sp(4)$ commutes with the standard action of $o(d-1,2)$. Similarly to the case of a scalar particle considered above it then follows that the BRST operator (4.14) is invariant under the choice of the local frame of $\boldsymbol{\mathcal{V}}(\mathcal{X}_0)$.

Taking as representation space $\Gamma(\boldsymbol{\mathcal{H}}^{\mathrm{T}})$, the space of "functions" in variables $x, Y, a^\dagger$, $c^\dagger, c_0, b^\dagger,\ \xi, \mu, \nu, \theta^\mu$ which are formal power series in the variables Y^A, smooth functions in x, and polynomials in $a^{\dagger A}$ and ghost variables, the quantum system described by Ω^{T} is the parent system for free higher spin gauge fields on AdS. The action of the BRST operator (4.14) on a generic state of $\phi \in \Gamma(\boldsymbol{\mathcal{H}}^{\mathrm{T}})$ takes the form

$$\Omega^{\mathrm{T}}\phi = (\nabla + \sigma + \bar{\Omega})\phi\,, \qquad (4.19)$$

where

$$\nabla = \boldsymbol{d} - \omega_A^B Y^A \frac{\partial}{\partial Y^B} - \omega_A^B a^{\dagger A}\frac{\partial}{\partial a^{\dagger B}}\,, \qquad \sigma = -\theta^\mu e_\mu^A \frac{\partial}{\partial Y^A}\,, \qquad (4.20)$$

and

$$\begin{aligned}\bar{\Omega} = {}& c_0\Box + c^\dagger S + S^\dagger \frac{\partial}{\partial b^\dagger} + \xi T + \mu(h-2) + \nu \bar{S}^\dagger - c^\dagger \frac{\partial}{\partial b^\dagger}\frac{\partial}{\partial c_0} - 2\xi \frac{\partial}{\partial b^\dagger}\frac{\partial}{\partial c^\dagger} - \\ & - 2\mu c_0 \frac{\partial}{\partial c_0} + 2\mu b^\dagger \frac{\partial}{\partial b^\dagger} + 2\mu\xi\frac{\partial}{\partial \xi} + 2\nu c_0 \frac{\partial}{\partial c^\dagger} + 2\mu\nu\frac{\partial}{\partial \nu} + \nu c^\dagger \frac{\partial}{\partial \xi} + \nu \frac{\partial}{\partial b^\dagger}\frac{\partial}{\partial \mu}\,. \qquad (4.21)\end{aligned}$$

The operators $\Box, T, S, \bar{S}^\dagger, S^\dagger, h$ are given by (4.15) with P_A, a_A replaced with $-\frac{\partial}{\partial Y^A}$ and $\frac{\partial}{\partial a^{\dagger A}}$ respectively. By various consistent reductions of the parent system $(\Omega^{\mathrm{T}}, \Gamma(\boldsymbol{\mathcal{H}}^{\mathrm{T}}))$, one can reach the original and the unfolded descriptions as well as some "intermediate" formulations which can be interesting in their own right.

4.3 Reduction to standard description

As a first step, a new intermediate reduction could turn out to be useful. It preserves the simple algebraic structure of the parent theory and involves the $d+1$ oscillators while eliminating all the Y-variables but one.

JHEP08(2006)013

4.3.1 Tensor fields in embedding space

The reduction consists in the elimination of Y^a, θ^μ. In this case, $\Gamma(\boldsymbol{\mathcal{E}})$ is the space of functions in x with values in formal power series in $Y^{(d)}$ and polynomials in $a^{\dagger A}$, c_0, $c^\dagger$, $b^\dagger$, μ, ν. We again choose $V^A = l\delta^A_{(d)}$ and set $Y^{(d)} = lz$, $a^{\dagger(d)} = lw$. Let $\partial_a = e^\mu_a \frac{\partial}{\partial x^\mu}$, $\omega_{a}{}^{B}{}_{C} = e^\mu_a \omega_\mu{}^B{}_C$ and

$$\mathcal{D}_a = \partial_a - \omega_a{}^B{}_C a^{\dagger C} \frac{\partial}{\partial a^{\dagger B}} = \partial_a - \omega_a{}^b{}_c a^{\dagger c} \frac{\partial}{\partial a^{\dagger b}} - w\frac{\partial}{\partial a^{\dagger a}} - \frac{a^\dagger_a}{l^2}\frac{\partial}{\partial w}\,, \tag{4.22}$$

so that $[\mathcal{D}_a, \mathcal{D}_b] = (\omega^c_{ab} - \omega^c_{ba})\mathcal{D}_c$. We show in appendix C.1 that the reduced BRST differential is given by

$$\begin{aligned}\widetilde{\Omega} = c_0\widetilde{\Box} + \widetilde{S}^\dagger\frac{\partial}{\partial b^\dagger} + c^\dagger\widetilde{S} + \xi\widetilde{T} + \mu\widetilde{h} + \nu\widetilde{\bar{S}}^\dagger - c^\dagger\frac{\partial}{\partial b^\dagger}\frac{\partial}{\partial c_0} - 2\mu c_0\frac{\partial}{\partial c_0} -\\ - 2\mu\frac{\partial}{\partial b^\dagger}b^\dagger + 2\mu\xi\frac{\partial}{\partial \xi} + 2\mu\nu\frac{\partial}{\partial\nu} - 2\xi\frac{\partial}{\partial b^\dagger}\frac{\partial}{\partial c^\dagger} + 2\nu c_0\frac{\partial}{\partial c^\dagger} + \nu c^\dagger\frac{\partial}{\partial\xi} + \nu\frac{\partial}{\partial b^\dagger}\frac{\partial}{\partial\mu}\,,\end{aligned} \tag{4.23}$$

where

$$\widetilde{\Box} \;=\; \left(\frac{1}{1+z}\right)^2\eta^{ac}(\delta^b_c\mathcal{D}_a - \omega_a{}^b{}_c)\mathcal{D}_b - \frac{1}{l^2}\frac{\partial}{\partial z}\frac{\partial}{\partial z} - \frac{d}{l^2}\frac{1}{1+z}\frac{\partial}{\partial z}\,, \tag{4.24}$$

$$\widetilde{S}^\dagger \;=\; \frac{1}{1+z}a^{\dagger a}\mathcal{D}_a + w\frac{\partial}{\partial z}\,, \tag{4.25}$$

$$\widetilde{S} \;=\; \frac{1}{1+z}\frac{\partial}{\partial a^\dagger_a}\mathcal{D}_a - \frac{1}{l^2}\frac{\partial}{\partial w}\frac{\partial}{\partial z}\,, \tag{4.26}$$

$$\widetilde{h} \;=\; a^{\dagger A}\frac{\partial}{\partial a^{\dagger A}} - (1+z)\frac{\partial}{\partial z}\,, \tag{4.27}$$

$$\widetilde{\bar{S}}^\dagger \;=\; (1+z)\frac{\partial}{\partial w}\,, \tag{4.28}$$

$$\widetilde{T} \;=\; \frac{\partial}{\partial a^\dagger_A}\frac{\partial}{\partial a^{\dagger A}}\,, \tag{4.29}$$

and the operators $\widetilde{\Box}$, $\widetilde{S}$, $\widetilde{S}^\dagger$, $\widetilde{T}$, $\widetilde{h}$, $\widetilde{\bar{S}}^\dagger$ satisfy the same subalgebra (4.16) of $sp(4)$ as the corresponding untiled operators.

One can reduce further by eliminating the dependence on ξ. This can be done consistently by restricting the remaining states, respectively the string fields, to be annihilated by $\widetilde{\mathcal{T}}_0 = \frac{\partial}{\partial a^\dagger_A}\frac{\partial}{\partial a^{\dagger A}} - 2\frac{\partial}{\partial b^\dagger}\frac{\partial}{\partial c^\dagger}$ and dropping all terms in the BRST operator that involve ξ or $\frac{\partial}{\partial\xi}$. Indeed, by choosing as a degree minus the homogeneity in ξ, the lowest part of the BRST operator (4.23) is $\widetilde{\Omega}_{-1} = \xi\widetilde{\mathcal{T}}_0$. Its cohomology is concentrated in degree 0, the ξ independent part, and described by states annihilated by $\widetilde{\mathcal{T}}_0$. It then follows directly from Proposition A.1 that the reduced BRST differential is given by $\widetilde{\Omega}_{\xi=0=\pi}$ restricted to the subspace of ξ-independent and $\widetilde{\mathcal{T}}_0$-traceless elements. Here, $\widetilde{\Omega}_{\xi=0=\pi}$ denotes the BRST operator $\widetilde{\Omega}$ with all ξ, $\frac{\partial}{\partial\xi}$-dependent terms dropped.

Note that one can consider the BRST operator $\widetilde{\Omega}_{\xi=0=\pi}$ as an operator acting in the subspace of ξ-independent elements that are not necessarily annihilated by $\widetilde{\mathcal{T}}_0$. However, this operator is not strictly nilpotent anymore, nor does it commute with $\widetilde{\mathcal{T}}_0$. More precisely, it satisfies $[\widetilde{\mathcal{T}}_0, \widetilde{\Omega}_{\xi=0=\pi}] = O\widetilde{\mathcal{T}}_0$, $\widetilde{\Omega}^2_{\xi=0=\pi} = P\widetilde{\mathcal{T}}_0$ for some operators O, P, as it should for $\widetilde{\Omega}_{\xi=0=\pi}$ to be well-defined and nilpotent on the $\widetilde{\mathcal{T}}_0$-traceless subspace.

Contrary to the case of higher spin gauge fields in flat space, one can thus not directly remove the trace constraint at the level of the parent theory or the intermediate reduction by simply imposing it on the states and the string fields and dropping $\xi, \frac{\partial}{\partial\xi}$ dependent terms in the BRST operator. The reason is that the commutator $[S, \bar{S}^\dagger] = T$ of operators S and $\bar{S}^\dagger$ entering the BRST operator produces T, and similarly for the tiled operators.

4.3.2 Tensor fields on AdS

We now consider the reduction to the standard BRST description in intrinsic coordinates, i.e., with z, w, μ, ν eliminated. The system is described by the space $\Gamma(\boldsymbol{\mathcal{E}})$ of functions in x^μ taking values in polynomials in $a^{\dagger a}, c_0, c^\dagger, b^\dagger, \xi$. In this case, we show in the appendix that the BRST operator reduces to

$$\widehat{\Omega} = c_0\widehat{\Box} + c^\dagger\widehat{S} + \widehat{S}^\dagger\frac{\partial}{\partial b^\dagger} - c^\dagger\frac{\partial}{\partial b^\dagger}\frac{\partial}{\partial c_0} + \xi\widehat{\mathcal{T}}\,, \tag{4.30}$$

where

$$\begin{aligned}
\widehat{\Box} &= \eta^{ac}(\delta^b_c D_a - \omega^{\ b}_{a\ c})D_b + \frac{1}{l^2}\big(N_{a^{\dagger a}} + 2c^\dagger\frac{\partial}{\partial\xi}a^{\dagger a}\nabla_a + \\
&\quad + (3-d-N_{a^{\dagger a}} - 2N_{b^\dagger} - 2N_\xi)(N_{a^{\dagger a}} - 2 + 2N_{b^\dagger} + 2N_\xi)\big)\,,\\
\widehat{S} &= \frac{\partial}{\partial a^\dagger_a}D_a + \frac{1}{l^2}\big(2c_0\frac{\partial}{\partial c^\dagger}(2N_{a^{\dagger a}} + d - 3 + 2N_{b^\dagger} + 2N_\xi)\big)\,,\\
\widehat{S}^\dagger &= a^{\dagger a}D_a + \frac{1}{l^2}a^{\dagger a}a^\dagger_a(c^\dagger\frac{\partial}{\partial\xi} + 2c_0\frac{\partial}{\partial c^\dagger})\,,\\
\widehat{\mathcal{T}} &= \frac{\partial}{\partial a^\dagger_a}\frac{\partial}{\partial a^{\dagger a}} - 2\frac{\partial}{\partial b^\dagger}\frac{\partial}{\partial c^\dagger} - \frac{1}{l^2}2c_0(1-2N_{c^\dagger})\frac{\partial}{\partial\xi}\,,
\end{aligned} \tag{4.31}$$

with $N_{Z^i} = Z^i\frac{\partial}{\partial Z^i}$ for any variables Z^i and

$$D_a = \partial_a - \omega^{\ b}_{a\ c}a^{\dagger c}\frac{\partial}{\partial a^{\dagger b}},\ [D_a, D_b] = (\omega^c_{ab} - \omega^c_{ba})D_c - \frac{1}{l^2}(a^\dagger_a\frac{\partial}{\partial a^{\dagger b}} - a^\dagger_b\frac{\partial}{\partial a^{\dagger a}})\,. \tag{4.32}$$

Finally, one can reduce further by eliminating the dependence on ξ. Exactly the same reasoning as for the analogous reduction in the previous subsection shows that this can be done consistently by restricting the remaining states, respectively the string fields, to be annihilated by $\widehat{\mathcal{T}}_0 = \frac{\partial}{\partial a^\dagger_a}\frac{\partial}{\partial a^{\dagger a}} - 2\frac{\partial}{\partial b^\dagger}\frac{\partial}{\partial c^\dagger}$ and dropping all terms in the BRST operator that involve ξ or $\frac{\partial}{\partial\xi}$. We denote by $\widehat{\Omega}_{\xi=0=\pi}$ the BRST operators $\widehat{\Omega}$ with all the $\xi, \frac{\partial}{\partial\xi}$ dependent terms dropped.

As before, one can consider $\widehat{\Omega}_{\xi=0=\pi}$ as an extension of the reduced BRST operator from the subspace of ξ-independent and $\widehat{\mathcal{T}}_0$-traceless elements to that of ξ-independent, but not necessarily $\widehat{\mathcal{T}}_0$-traceless ones. A more convenient extension, however, turns out to be

$$\Omega_{\rm mod} = \widehat{\Omega}_{\xi=0=\pi} + \frac{c_0}{l^2}(a^{\dagger a}a^\dagger_a + 4c^\dagger b^\dagger)\widehat{\mathcal{T}}_0\,, \tag{4.33}$$

because $\Omega_{\rm mod}$ is strictly nilpotent and commutes with $\widehat{\mathcal{T}}_0$.

Up to conventions and an overall sign, $\Omega_{\rm mod}$ coincides with the hermitian BRST operator constructed in this context in [36] (see also [37]). By constraining the string field to be annihilated by $\widehat{\mathcal{T}}_0$ and assuming the appropriate reality condition, the master action for higher spin fields on AdS is given by (2.7), using either $\Omega_{\rm mod}$ or $\widehat{\Omega}_{\xi=0=\pi}$. More details on master actions of this type can be found for instance in [40, 25, 50].

It then follows from Subsection 2.3 that all the formulations of higher spin gauge fields on AdS described in this paper are Lagrangian by using suitable generalized auxiliary fields.

4.4 Reduction to unfolded form

The reduction to the unfolded form is performed by reducing to the cohomology of $\bar{\Omega}$ given by (4.21). As in the flat case [25], we do this reduction in several steps.

4.4.1 Reducing to totally traceless fields

First we reduce the parent theory to a theory with totally traceless fields. This is achieved by taking as a degree minus the homogeneity in $c_0, c^\dagger, \xi$ and reducing to the cohomology of the part of $\bar{\Omega}$ in lowest degree -1,

$$\Omega_{\rm trace} = c_0\Box + c^\dagger S + \xi T\,. \tag{4.34}$$

The dimension of the space in which the trace is taken is $d+1$, the dimension of the embedding space. In $d+1 \geqslant 3$, the cohomology of $\Omega_{\rm trace}$ in $\mathcal{H}^{\rm T}$, the space of formal power series in variables Y^A with coefficients in polynomials in $a^{\dagger A}$ and ghost variables, is given by [25]:

$$\begin{aligned} &H^0(\Omega_{\rm trace}, \mathcal{H}^{\rm T}) \cong \widetilde{\mathcal{E}} = \{\phi \in \mathcal{H}^{\rm T} : \Box\phi = S\phi = T\phi = 0,\ \deg(\phi) = 0\}\,,\\ &H^n(\Omega_{\rm trace}, \mathcal{H}^{\rm T}) = 0 \qquad n \neq 0\,. \end{aligned} \tag{4.35}$$

Since the cohomology is concentrated in one degree, one immediately arrives at:

Proposition 4.1. *The parent system $(\Omega^{\rm T}, \Gamma(\boldsymbol{\mathcal{H}^{\rm T}}))$ can be consistently reduced to the system $(\widetilde{\Omega}^{\rm T}, \Gamma(\boldsymbol{\widetilde{\mathcal{E}}}))$ with $\widetilde{\Omega}^{\rm T} = \nabla + \sigma + \widetilde{\Omega}$ and*

$$\widetilde{\Omega} \;=\; S^\dagger\frac{\partial}{\partial b^\dagger} + \mu(h-2) + \nu\bar{S}^\dagger + 2\mu b^\dagger\frac{\partial}{\partial b^\dagger} + 2\mu\nu\frac{\partial}{\partial\nu} + \nu\frac{\partial}{\partial b^\dagger}\frac{\partial}{\partial\mu}\,. \tag{4.36}$$

The associated field theory is described by the physical fields F, H, G, A taking values in $Y, a^\dagger$-dependent traceless elements and entering the ghost number zero component of the string field

$$\widetilde{\Psi}^{(0)} = F + \mu b^\dagger H + \nu b^\dagger G + b^\dagger\theta^\mu A_\mu\,. \tag{4.37}$$

In terms of component fields, the equations of motion $\widetilde{\Omega}^{\rm T}\widetilde{\Psi}^{(0)} = 0$ read as

$$\begin{aligned} &DA = 0, \qquad DF + S^\dagger A = 0, \qquad DH + hA = 0, \qquad DG + \bar{S}^\dagger A = 0,\\ &(h-2)F - S^\dagger H = 0, \qquad \bar{S}^\dagger F + H - S^\dagger G = 0, \qquad (h+2)G - \bar{S}^\dagger H = 0\,, \end{aligned} \tag{4.38}$$

where $D = \nabla + \sigma$. The gauge symmetries are determined by $\delta\widetilde{\Psi}^{(0)} = \widetilde{\Omega}^{\mathrm{T}}\widetilde{\Psi}^{(1)}$ with component fields in $\widetilde{\Psi}^{(1)}$ to be replaced with gauge parameters.

The next step is to reduce to the cohomology of the BRST operator (4.36). This operator corresponds to the standard Chevalley-Eilenberg differential associated with the Lie algebra $sl(2)$ in the given representation.

If V^A were vanishing, the Lie algebra would act homogeneously in $Y, a^\dagger$ and the representation space would split into the direct sum of finite-dimensional irreducible representations. In this case, the cohomology is well known and given by the Lie algebra invariants in ghost numbers $-1, 1$. In our case, however, the operators act inhomogeneously so that infinite-dimensional representations have to be taken into account. In particular, there can be nontrivial cohomology classes associated with elements which are not polynomial but are formal power series in Y. It is again instructive to split the reduction to the cohomology of $\widetilde{\Omega}$ into two steps.

4.4.2 First step: reduction to the intermediate system

Taking as a degree minus the homogeneity in ghost variables μ, ν, one arrives at the decomposition $\widetilde{\Omega} = \widetilde{\Omega}_{-1} + \widetilde{\Omega}_0$, where

$$\widetilde{\Omega}_{-1} = \mu(h-2) + \nu\bar{S}^\dagger + 2\mu b^\dagger\frac{\partial}{\partial b^\dagger} + 2\mu\nu\frac{\partial}{\partial\nu}\,, \qquad \widetilde{\Omega}_0 = S^\dagger\frac{\partial}{\partial b^\dagger} + \nu\frac{\partial}{\partial b^\dagger}\frac{\partial}{\partial\mu}\,. \tag{4.39}$$

In order to carry out the first step of the reduction, we need:

Proposition 4.2. *The cohomology of $\widetilde{\Omega}_{-1}$ in $\widetilde{\mathcal{E}}$ is given by*

$$H^0(\widetilde{\Omega}_{-1}, \widetilde{\mathcal{E}}) = \widehat{\mathcal{E}}\,, \qquad H^n(\widetilde{\Omega}_{-1}, \widetilde{\mathcal{E}}) = 0 \quad n \neq 0, \tag{4.40}$$

where $\widehat{\mathcal{E}} \subset \widetilde{\mathcal{E}}$ is the subspace of μ, ν-independent elements satisfying

$$\left(h - 2 + 2b^\dagger\frac{\partial}{\partial b^\dagger}\right)\phi = 0\,, \qquad \bar{S}^\dagger\phi = 0\,. \tag{4.41}$$

In degree zero, the statement is trivial. That the cohomology of $\widetilde{\Omega}_{-1}$ vanishes in nonzero degree is shown in appendix C.2. Because the cohomology of $\widetilde{\Omega}_{-1}$ is concentrated in one degree, the reduction is straightforward:

Proposition 4.3. *The system $(\widetilde{\Omega}^{\mathrm{T}}, \Gamma(\widetilde{\boldsymbol{\mathcal{E}}}))$ can be consistently reduced to the* intermediate system $(\widehat{\Omega}^{\mathrm{T}}, \Gamma(\widehat{\boldsymbol{\mathcal{E}}}))$ *where $\widehat{\mathcal{E}}$ is described by* (4.41) *and*

$$\widehat{\Omega}^{\mathrm{T}} = \nabla + \sigma + S^\dagger\frac{\partial}{\partial b^\dagger}\,. \tag{4.42}$$

Note that $\nabla + \sigma$ commutes with $\bar{S}^\dagger$ and $h - 2 + 2b^\dagger\frac{\partial}{\partial b^\dagger}$. Similarly, $S^\dagger\frac{\partial}{\partial b^\dagger}$ preserves $\widehat{\mathcal{E}}$ and therefore projectors are not needed in (4.42).

The equations of motion of the associated free field theory have the form

$$(\nabla + \sigma)\widehat{A} = 0\,, \qquad (\nabla + \sigma)\widehat{F} = -S^\dagger\widehat{A}\,. \tag{4.43}$$

Here $\widehat{A} = \theta^\mu \widehat{A}_\mu(x; Y, a^\dagger)$ and $\widehat{F}(x; Y, a^\dagger)$ are respectively the 1-form and the 0-form physical fields entering the string field associated with $\widehat{\mathcal{E}}$:

$$\widehat{\Psi}^{(0)} = \widehat{F} + b^\dagger \theta^\mu \widehat{A}_\mu \,. \tag{4.44}$$

The gauge transformations are given by

$$\delta \widehat{A} = (\nabla + \sigma)\widehat{\lambda}\,, \qquad \delta \widehat{F} = -S^\dagger \widehat{\lambda}\,, \tag{4.45}$$

where the gauge parameter $\widehat{\lambda}(x; Y, a^\dagger)$ replaces the fields at ghost number 1 and therefore satisfies $h\widehat{\lambda} = \bar{S}^\dagger \widehat{\lambda} = 0$.

This representation of higher spin gauge fields generalizes the so-called *intermediate form* identified in [25] to the case of a constant curvature background.

4.4.3 Second step: reduction to the unfolded form

In the next step, we take as a degree the homogeneity in $b^\dagger$ so that $\widehat{\Omega}_{-1} = S^\dagger \frac{\partial}{\partial b^\dagger}$. To reduce the system, we need to compute the cohomology of $\widehat{\Omega}_{-1}$ in the space $\widehat{\mathcal{E}}$ of completely traceless formal power series in Y with coefficients in polynomials in $a^\dagger, b^\dagger, \theta^\mu$ satisfying (4.41).

In degree 1, the coboundary condition is trivial while the cocycle condition gives $S^\dagger \phi = 0$. This condition is homogeneous in variables $Y, a^\dagger$ and implies that a homogeneous component of ϕ has more $a^\dagger$ than Y variables. Together with the condition $h\phi = 0$, this implies that ϕ is polynomial in Y as well since otherwise the condition $h\phi = 0$ can be satisfied only by a formal power series in z, which contradicts $S^\dagger \phi = 0$. Therefore any solution can be decomposed into homogeneous solutions in $a^{\dagger A}$ and $Y^A + V^A$. These solutions are described by traceless rectangular Young tableaux.

In degree 0, the cocycle condition is trivial while the coboundary tells us that $\phi \sim \phi + S^\dagger A$ for a traceless A satisfying $hA = \bar{S}^\dagger A = 0$. In fact in each equivalence class there exists a unique representative satisfying $V^A \frac{\partial}{\partial a^{\dagger A}} \phi = 0$. Because the statement does not depend on the choice of local frame it is enough to show this in the frame where $V^A = l\delta^A_{(d)}$. As usually we use notations: $lz = lz^1 = Y^{(d)}$, $lw = lz^2 = a^{\dagger(d)}$, and $y^a = Y^a$. We need the following:

Proposition 4.4. *For any z^α independent $\phi_0 \in \widetilde{\mathcal{E}}$ and an arbitrary number m, there exists a unique solution $\phi \in \widetilde{\mathcal{E}}$ satisfying the equation:*

$$(h - m)\phi = 0\,, \quad \bar{S}^\dagger \phi = 0\,, \tag{4.46}$$

and the boundary condition $\mathcal{P}(\phi|_{z^\alpha=0}) = \phi_0$, where $\mathcal{P}$ is the projector to the subspace of totally traceless elements in the space of z^α independent elements.

The proof is completely similar to that of Proposition C.2 given in appendix C.2. The proposition determines a map K_m that sends a traceless z^α-independent element ϕ_0 to a traceless element ϕ satisfying (4.46) and $\mathcal{P}(\phi|_{z^\alpha=0}) = \phi_0$.

Furthermore, if $\phi = K_2\phi_0$ one finds that $\mathcal{P}((S^\dagger \phi)|_{z^\alpha=0}) = S_0^\dagger \phi_0$ where

$$S_0^\dagger = a^{\dagger b} \frac{\partial}{\partial Y^b}\,, \qquad h_0 = a^{\dagger b} \frac{\partial}{\partial a^{\dagger b}} - y^b \frac{\partial}{\partial y^b}\,, \qquad \bar{S}_0^\dagger = y^b \frac{\partial}{\partial a^{\dagger b}}\,, \tag{4.47}$$

form the standard representation of $sl(2)$ in the space of z, w-independent elements. It then follows that for any traceless z, w-independent ϕ_0 there exists a unique element ϕ_0' such that $\bar{S}_0^\dagger \phi_0' = 0$ and $\phi_0' = \phi_0 + S_0^\dagger A_0$ for some traceless z, w-independent element A_0. Indeed, in each irreducible component any element can be uniquely represented as a linear combination of the element annihilated by $\bar{S}_0^\dagger$ (i.e., proportional to the lowest weight vector) and the element in the image of $S_0^\dagger$. Using Proposition 4.4 one then finds a unique A satisfying $\bar{S}^\dagger A = hA = 0$ and $\mathcal{P}(A|_{z^\alpha=0}) = A_0$ and finds that $\phi' = \phi + S^\dagger A$ satisfies $\mathcal{P}(\phi'|_{z^\alpha=0}) = \phi_0'$. Finally, one observes that $\phi' = \mathcal{K}_m \phi_0'$ does not depend on w for any m provided $\bar{S}^\dagger \phi_0 = 0$. Using $\frac{\partial}{\partial w} = \frac{1}{l^2} V^A \frac{\partial}{\partial a^{\dagger A}}$ one then concludes that the unique representative of a cohomology class in degree 0 can be assumed to satisfy

$$(h-2)\phi = 0\,, \qquad \bar{S}^\dagger \phi = 0\,, \qquad V^A \frac{\partial}{\partial a^{\dagger A}} \phi = 0\,. \tag{4.48}$$

The decomposition of $\widehat{\mathcal{E}}$ then reads:

$$\widehat{\mathcal{E}} = \mathcal{E} \oplus \mathcal{F} \oplus \mathcal{G}\,, \qquad \mathcal{E} = \mathcal{E}_0 \oplus \mathcal{E}_1\,, \tag{4.49}$$

where $\mathcal{E}_1$ is the subspace of elements of the form $b^\dagger \chi$ with χ satisfying $h\phi = S^\dagger \phi = \bar{S}^\dagger \phi = 0$, $\mathcal{E}_0$ is determined by (4.48), $\mathcal{G} = \operatorname{Im} S^\dagger \frac{\partial}{\partial b^\dagger}$ in $\widehat{\mathcal{E}}$, and $\mathcal{F}$ is a complementary subspace.

We are now in the position to compute the reduced differential

$$\Omega^{\mathrm{unf}} = \overset{\boldsymbol{\mathcal{E}\mathcal{E}}}{D} - \overset{\boldsymbol{\mathcal{E}\mathcal{F}}}{D}\,\rho\,\overset{\boldsymbol{\mathcal{G}\mathcal{E}}}{D}\,, \qquad D = \nabla + \sigma\,, \tag{4.50}$$

where $\rho : \Gamma(\boldsymbol{\mathcal{G}}) \to \Gamma(\boldsymbol{\mathcal{F}})$ is the inverse to $\widehat{\Omega}_{-1}$. Note that because the cohomology is concentrated in degree 0 and 1, the higher order terms in $(\overset{\boldsymbol{\mathcal{G}\mathcal{F}}}{\Omega})^{-1} = \rho + \cdots$ cannot contribute and therefore there is only one additional term besides $\overset{\boldsymbol{\mathcal{E}\mathcal{E}}}{\Omega}$ in (4.50).

Proposition 4.5. *The system* $(\widehat{\Omega}^{\mathrm{T}}, \Gamma(\widehat{\boldsymbol{\mathcal{E}}}))$ *can be consistently reduced to the unfolded system* $(\Omega^{\mathrm{unf}}, \Gamma(\boldsymbol{\mathcal{E}}))$ *with*

$$\Omega^{\mathrm{unf}} = \nabla + \sigma - b^\dagger \sigma \bar{\sigma} \mathcal{P}_R\,, \qquad \bar{\sigma} = \theta^\mu e_\mu^A \frac{\partial}{\partial a^{\dagger A}}\,. \tag{4.51}$$

Here $\mathcal{P}_R$ denotes the projector from $\mathcal{E}_0$ to the space of traceless elements described by rectangular Young tableaux defined as follows: if $\phi = \phi^0 + \phi^1 + \cdots$ where $(a^{\dagger A} \frac{\partial}{\partial a^{\dagger A}} - Y^A \frac{\partial}{\partial Y^A})\phi^k = -k\phi^k$ then $\mathcal{P}_R \phi = \phi^0$.

Note that $\mathcal{P}_R$ is not a projector onto a subspace of $\mathcal{E}_0$. The proof of the proposition is again relegated to appendix C.2. Note that the last term in Ω^{unf} automatically belongs to $\mathcal{E}_1$ because it follows from $V^A \frac{\partial}{\partial a^{\dagger A}} \phi = 0$ that $V^A \frac{\partial}{\partial a^{\dagger A}} \mathcal{P}_R \phi = 0$ and therefore $(\bar{S}^\dagger - V^A \frac{\partial}{\partial a^{\dagger A}}) \mathcal{P}_R \phi = 0$. At the same time $(a^{\dagger A} \frac{\partial}{\partial a^{\dagger A}} - Y^A \frac{\partial}{\partial Y^A}) \mathcal{P}_R \phi = 0$. Because $\bar{S}^\dagger - V^A \frac{\partial}{\partial a^{\dagger A}}$, $h + V^A \frac{\partial}{\partial Y^A}$, and $S^\dagger$ form a standard presentation of $sl(2)$ on the space of $a^{\dagger A}, Y^A$-dependent elements, $\mathcal{P}_R \phi$ is $sl(2)$ invariant and therefore $S^\dagger \mathcal{P}_R \phi = 0$. Using $\bar{S}^\dagger \mathcal{P}_R \phi = 0$ one concludes that $h \mathcal{P}_R \phi = 0$ as well so that $b^\dagger \sigma \bar{\sigma} \mathcal{P}_R \phi \in \mathcal{E}_1$.

The equations of motion determined by Ω^{unf} take the form $\Omega^{\mathrm{unf}} \Psi^{\mathrm{unf}(0)} = 0$ and explicitly read as

$$(\nabla + \sigma) F = 0\,, \qquad (\nabla + \sigma) A + \sigma \bar{\sigma} \mathcal{P}_R F = 0\,, \tag{4.52}$$

JHEP08(2006)013

where the physical fields A, F enter the zero-ghost-number component of the string field:

$$\Psi^{\mathrm{unf}(0)} = F(x; Y, a^\dagger) + b^\dagger \theta^\mu A_\mu(x; Y, a^\dagger)\,. \tag{4.53}$$

Under a gauge transformation, F is invariant while $\delta A = (\nabla + \sigma)\lambda^{\mathrm{unf}}$ where $\lambda^{\mathrm{unf}}(x; Y, a)$ is a gauge parameter with values in the subspace of elements annihilated by $S^\dagger, \bar{S}^\dagger, h$.

Finally, we will rewrite the unfolded system in terms of z^α-independent fields in order to arrive at the formulation in terms of intrinsic coordinates. To this end we assume that the local frame is chosen such that $V^A = l\delta^A_{(d)}$ and as usually $lw = a^{\dagger(d)}$, $lz = Y^{(d)}$, and $y^a = Y^a$. It follows from Proposition 4.4 that any element of $\widehat{\mathcal{E}}$ is uniquely determined by the traceless part of its z^α-independent part. It is useful to describe subspaces $\mathcal{E}_0$ and $\mathcal{E}_1$ in this way. For $\mathcal{E}_1$, equations $h\chi = \bar{S}^\dagger\chi = S^\dagger\chi = 0$ imply that $\chi_0 = \mathcal{P}\chi|_{z^\alpha=0}$ satisfy $S_0^\dagger\chi_0 = 0$ from which it follows that $\mathcal{E}_1$ is isomorphic to the space $\bar{\mathcal{E}}_1$ of z^α-independent and linear in $b^\dagger$ traceless elements described by two row Young tableaux for which the number of indexes contracted with $a^{\dagger b}$ is bigger than that contracted with y^b. The isomorphism is just $\mathcal{K}_0$ considered as a map from $\bar{\mathcal{E}}_1$ to $\mathcal{E}_1$.

For $\mathcal{E}_0$, equations (4.48) imply that $\phi_0 = \phi|_{z^\alpha=0}$ satisfy $\bar{S}_0^\dagger\phi_0 = 0$ and therefore the space $\mathcal{E}_0$ is isomorphic to the space $\bar{\mathcal{E}}_0$ of $b^\dagger, z^\alpha$-independent traceless elements described by two row Young tableaux for which the number of indexes contracted with y^b is bigger than that contracted with $a^{\dagger b}$. The isomorphism is just $\mathcal{K}_2$ considered as a map from $\bar{\mathcal{E}}_0$ to $\mathcal{E}_0$.

This shows that the field content matches that of the unfolded form of higher spin gauge fields in terms of intrinsic coordinates [26, 38]. It is instructive to write down the structure of a representative ϕ of $\mathcal{E}_0$ satisfying (4.48) in terms of its z^α-independent traceless part ϕ_0. One finds

$$\phi = \mathcal{K}_2\phi_0 = \frac{1}{(1+z)^{-h_0+2}}\left(\phi_0 + \frac{(n-s)(n-s+1)}{2l^2(d+2n-4)}(y^a y_a)\phi_0 + \cdots\right) \tag{4.54}$$

where the ratio is understood as a formal power series and $\ldots$ denote terms proportional to $(y^a y_a)^k\phi_0$ with $k \geqslant 2$.

As for elements from $\mathcal{E}_1$, let χ_0 be a traceless z^α-independent element satisfying $S_0^\dagger\chi_0 = 0$. Then its representative in $\mathcal{E}_1$ has the form

$$\begin{aligned}\chi = \mathcal{K}_0\chi_0 = (z+1)^{h_0+N_w}\big(\chi_0 - w\bar{S}^\dagger\chi_0 + \cdots +\\ + (y^a y_a)(\ldots) + (y^a a_a^\dagger)(\ldots) + (a^{\dagger a} a_a^\dagger)(\ldots)\big)\,,\end{aligned} \tag{4.55}$$

where $N_w = w\frac{\partial}{\partial w}$, $\ldots$ denote terms proportional to $(w\bar{S}_0^\dagger)^k\chi_0$ with $k \geqslant 2$, and terms in parenthesis denote some polynomials.

In addition we need the explicit expression for $D = \nabla + \sigma$ in the frame where $V^A = l\delta^A_{(d)}$:

$$D = \nabla_0 + \sigma - \frac{1}{l^2}e^a y_a \frac{\partial}{\partial z} - \frac{1}{l^2}e^a a_a^\dagger \frac{\partial}{\partial w} - e^a z\frac{\partial}{\partial y^a} - e^a w\frac{\partial}{\partial a^{\dagger a}}\,, \tag{4.56}$$

where ∇_0 denotes the d-dimensional covariant derivative, i.e.,

$$\nabla_0 = \boldsymbol{d} - \omega^b{}_a y^a \frac{\partial}{\partial y^b} - \omega^b{}_a a^{\dagger a}\frac{\partial}{\partial a^{\dagger b}}\,. \tag{4.57}$$

Proposition 4.6. *In terms of z, w independent elements, the unfolded system takes the form*

$$\Omega^{\mathrm{unf}}(\boldsymbol{\phi}_0 + b^\dagger \boldsymbol{\chi}_0) = D_{\bar{\mathcal{E}}_0}\boldsymbol{\phi}_0 - b^\dagger D_{\bar{\mathcal{E}}_1}\boldsymbol{\chi}_0 - b^\dagger\sigma\bar{\sigma}\mathcal{P}^0_R\boldsymbol{\phi}_0 \tag{4.58}$$

where $\boldsymbol{\phi}_0 \in \Gamma(\bar{\boldsymbol{\mathcal{E}}}_{\mathbf{0}})$, $b^\dagger\boldsymbol{\chi}_0 \in \Gamma(\bar{\boldsymbol{\mathcal{E}}}_{\mathbf{1}})$, and $\mathcal{P}^0_R$ denotes the projector to the subspace of elements in $\bar{\mathcal{E}}_0$ described by rectangular Young tableaux. Furthermore, if $n = y^a\frac{\partial}{\partial y^a}$ and $s = a^{\dagger a}\frac{\partial}{\partial a^{\dagger a}}$

$$\begin{aligned} D_{\bar{\mathcal{E}}_0}\boldsymbol{\phi}_0 = \nabla_0\boldsymbol{\phi}_0 + \sigma\boldsymbol{\phi}_0 + \frac{1}{n-s+2}S_0^\dagger\bar{\sigma}\boldsymbol{\phi}_0 \, + \\ + \, \frac{(n-s+1)(d+n+s-4)}{l^2(d+2n-2)}\mathcal{P}\left[e_a y^a\boldsymbol{\phi}_0\right], \end{aligned} \tag{4.59}$$

and

$$D_{\bar{\mathcal{E}}_1}\boldsymbol{\chi}_0 \;=\; \nabla_0\boldsymbol{\chi}_0 \,+\, \sigma\boldsymbol{\chi}_0 \,-\, \frac{(d+s+n-4)}{l^2(d+2n-4)}\mathcal{P}\left[(s-n+1)e^a y_a\boldsymbol{\chi}_0 - e^a a^\dagger_a\bar{S}^\dagger_0\boldsymbol{\chi}_0\right]. \tag{4.60}$$

Again, the proof is given in appendix C.2. Using (4.58), equations (4.52) take the form

$$D_{\bar{\mathcal{E}}_0}\bar{F} = 0\,, \qquad D_{\bar{\mathcal{E}}_1}\bar{A} + \sigma\bar{\sigma}\mathcal{P}^0_R\bar{F} = 0\,. \tag{4.61}$$

where $\bar{F} = \mathcal{P}(F|_{z^\alpha=0})$, $\bar{A} = \mathcal{P}(A|_{z^\alpha=0})$, and $\mathcal{P}^0_R$ is the projector to the subspace of elements described by rectangular Young tableaux. The fields entering $\bar{F}$ are gauge invariant while for those in $\bar{A}$ one gets $\delta_{\bar{\lambda}}\bar{A} = D_{\bar{\mathcal{E}}_1}\bar{\lambda}$ where the gauge parameter takes values in the subspace of z^α-independent traceless elements annihilated by $S_0^\dagger$. Up to conventions and normalization factors, these equations indeed coincide with the unfolded form of higher spin equations [38, 26] in AdS space.

Acknowledgments

The authors acknowledge useful discussions with G. Bonelli. M.G. wants to thank the International Solvay Institutes for hospitality. He is also grateful to K. Alkalaev, I. Tipunin, A. Semikhatov, I. Tyutin, and M. Vasiliev for useful discussions. The work of G.B. is supported in part by a “Pôle d'Attraction Interuniversitaire” (Belgium), by IISN-Belgium, convention 4.4505.86, by Proyectos FONDECYT 1970151 and 7960001 (Chile) and by the European Commission program MRTN-CT-2004-005104, in which this author is associated to V.U. Brussels. The work of M.G. was supported by the RFBR Grant 04-01-00303, the RFBR–JSPS Grant 05-01-02934YaF_a, and by the Grant LSS-1578.2003.2.

A. Reduction in homological terms and $\mathcal{D}$-modules

Proposition A.1. *Suppose $\boldsymbol{\mathcal{H}}$ to be equipped with an additional grading besides the ghost number,*

$$\boldsymbol{\mathcal{H}} = \bigoplus_{i\geqslant 0}\boldsymbol{\mathcal{H}}_i, \qquad \deg(\boldsymbol{\mathcal{H}}_i) = i, \tag{A.1}$$

JHEP08(2006)013

and let the BRST operator Ω have the form

$$\Omega = \Omega_{-1} + \Omega_0 + \sum_{i\geqslant 1} \Omega_i, \qquad \deg(\Omega_i) = i, \tag{A.2}$$

with $\Omega_i : \Gamma(\boldsymbol{\mathcal{H}})_j \to \Gamma(\boldsymbol{\mathcal{H}})_{i+j}$. If Ω_{-1} is a linear map of vector bundles (i.e. does not contain x-derivatives) then $H(\Omega_{-1}, \Gamma(\mathcal{H})) \cong \Gamma(\boldsymbol{\mathcal{E}})$ for some vector bundle $\boldsymbol{\mathcal{E}}$ and the system $(\Omega, \Gamma(\boldsymbol{\mathcal{H}}))$ can be consistently reduced to $(\widetilde{\Omega}, \Gamma(\boldsymbol{\mathcal{E}}))$ where the operator $\widetilde{\Omega}$ is the differential induced by Ω in the cohomology of Ω_{-1}.

Note that without loss of generality one can assume that $\boldsymbol{\mathcal{E}}$ is a subbundle in $\boldsymbol{\mathcal{H}}$. Moreover one can always find a decomposition $\boldsymbol{\mathcal{H}} = \boldsymbol{\mathcal{E}} \oplus \boldsymbol{\mathcal{G}} \oplus \boldsymbol{\mathcal{F}}$ where $\operatorname{Ker} \Omega_{-1} = \boldsymbol{\mathcal{E}} \oplus \boldsymbol{\mathcal{G}}$, $\boldsymbol{\mathcal{E}} \cong H(\Omega_{-1}, \boldsymbol{\mathcal{H}})$, $\boldsymbol{\mathcal{G}} = \operatorname{Im} \Omega_{-1}$, and $\boldsymbol{\mathcal{F}}$ is a complementary subbundle. Then $\overset{\boldsymbol{\mathcal{GF}}}{\Omega}$ is algebraically invertible and $\widetilde{\Omega}$ is given by (2.3). Note also that if the cohomology of Ω_{-1} is concentrated in one degree then $\widetilde{\Omega} = \Omega_0$ considered as acting in $\Gamma(\boldsymbol{\mathcal{E}})$. Note that Proposition A.1 is a slightly generalized version of the one in [25]. After choosing an adapted local frame, its proof reduces to that in [25]. In that reference, one can also find an explicit recursive construction for $\widetilde{\Omega}$.

Let us note that two systems $(\Omega, \Gamma(\boldsymbol{\mathcal{H}}))$ and $(\Omega', \Gamma(\boldsymbol{\mathcal{H}}'))$ are obviously equivalent when vector bundles $\boldsymbol{\mathcal{H}}$ and $\boldsymbol{\mathcal{H}}'$ are isomorphic and the isomorphism maps Ω into Ω'. At the level of associated field theories the respective theories are related by a field redefinition $\psi^A \to O^A_B(x)\psi^B$, where O^A_B are the components of the isomorphism map with respect to the local frames. This is a very restricted class of field redefinitions because it does not involve x-derivatives of fields. A more general class of equivalence relations is provided by allowing O^A_B to be algebraically invertible. Note that this notion of equivalence is completely natural from the quantum mechanical point of view because general similarity transformations in x-representation are allowed to be invertible operators containing x-derivatives.

In fact there exists an adequate language which allows for a more invariant formulation of first-quantized systems. This amounts to replacing the space of sections, which is a module over functions on $\mathcal{X}$, with the $\mathcal{D}$-module over the algebra of differential operators on $\mathcal{X}$. We now shortly describe how to formulate the basic notions in terms of $\mathcal{D}$-modules.

Let $\mathcal{D}_{\mathcal{X}}$ be the algebra of differential operators on $\mathcal{X}$ of (graded) finite order. Let also $\mathcal{D}(\boldsymbol{\mathcal{H}})$ be a right module over $\mathcal{D}_{\mathcal{X}}$ generated by the vector bundle $\boldsymbol{\mathcal{H}}$. Locally on $\mathcal{X}$, $\mathcal{D}(\boldsymbol{\mathcal{H}})$ is a free $\mathcal{D}_{\mathcal{X}}$-module generated by a local frame e_A. A general element of $\mathcal{D}(\boldsymbol{\mathcal{H}})$ can then be represented as $f = e_A f^A(x, \frac{\partial}{\partial x})$ where e_A is a local frame of $\boldsymbol{\mathcal{H}}$ and $f^A(x, \frac{\partial}{\partial x})$ are some differential operators. The action of linear differential operators on $\boldsymbol{\mathcal{H}}$ can be extended to $\mathcal{D}(\boldsymbol{\mathcal{H}})$ as follows: if $G(e_A \boldsymbol{\phi}^A) = e_B G^B_A \boldsymbol{\phi}^A$ for $\boldsymbol{\phi} \in \Gamma(\boldsymbol{\mathcal{H}})$ then for $f \in \mathcal{D}(\boldsymbol{\mathcal{H}})$ one has $Gf = G(e_A f^A) = e_A G^A_B \circ f^B$ where $f \circ \phi$ denotes the composition of differential operators, i.e., the associative product in the algebra of operators on $\mathcal{X}$. Note that this left action is compatible with the right module structure in the sense that it commutes with the right multiplication by differential operators.

It is also natural to choose a local frame e_A to be operator valued. For example, if e_A is a local frame of $\boldsymbol{\mathcal{H}}$ and $e'_A = e_B O^B_A(x, \frac{\partial}{\partial x})$, with O^A_B some algebraically invertible differential operator, then in the new frame the BRST operator Ω takes the form $\Omega e'_A = e'_B \circ \Omega'^B_A =$

$e'_B \circ (O^{-1})^B_C \circ \Omega^C_D \circ O^D_A$. If one associates fields ψ^A and ψ'^A to e_A and e'_A, the associated field theories determined by Ω and Ω' are related by a field redefinition $\psi^A = O^A_B \psi'^B$. In particular, two systems $(\Omega, \Gamma(\boldsymbol{\mathcal{H}}))$ and $(\Omega', \Gamma(\boldsymbol{\mathcal{H}}'))$ are isomorphic if $\mathcal{D}(\boldsymbol{\mathcal{H}}')$ and $\mathcal{D}(\boldsymbol{\mathcal{H}}))$ are isomorphic as right $\mathcal{D}_{\mathcal{X}}$-modules and the isomorphism maps Ω to Ω'. We also note that, because the action of Ω commutes with the right multiplication by a differential operators, the kernel, image, cohomology etc. of Ω are again $\mathcal{D}_{\mathcal{X}}$-modules, though not necessarily generated by vector bundles.

The approach just discussed is a BRST extension of the standard $\mathcal{D}$-module approach to partial differential equations (for a review see e.g. [51]). Note, however, that in the standard approach, left $\mathcal{D}$ modules are used which are in fact dual to the ones described above.

B. Particle on AdS: details of reductions

B.1 Standard description

Let us assume that we are in the frame where $V^A = l\delta^A_{(d)}$ and let $z = lY^{(d)}$ and $y^a = Y^a$. Following subsection 4.2 of [25], we choose as grading $Y^A \frac{\partial}{\partial Y^A} + 2c_0 \frac{\partial}{\partial c_0}$, so that the BRST operator (3.20) decomposes as $\Omega = \Omega_{-1} + \Omega_0$, where $\Omega_{-1} = -\theta^\mu e^a_\mu \frac{\partial}{\partial y^a} - \mu \frac{\partial}{\partial z}$, while

$$\Omega_0 = \theta^\mu (\frac{\partial}{\partial x^\mu} - \omega_{\mu}{}^{A}{}_{B} Y^B \frac{\partial}{\partial Y^A}) + c_0 \Box - \mu Y^A \frac{\partial}{\partial Y^A} + 2c_0 \eta \frac{\partial}{\partial c_0},$$

where $\Box = \frac{\partial}{\partial Y^A} \frac{\partial}{\partial Y_A}$. It follows that

$$\boldsymbol{\mathcal{H}}^{\mathrm{T}} = \boldsymbol{\mathcal{E}} \oplus \boldsymbol{\mathcal{F}} \oplus \boldsymbol{\mathcal{G}}$$

where $\boldsymbol{\mathcal{E}} \cong H(\Omega_{-1}, \boldsymbol{\mathcal{H}}^{\mathrm{T}})$. Introducing $\rho = -y^a e^\mu_a \frac{\partial}{\partial \theta^\mu} - z \frac{\partial}{\partial \mu}$ and $N = Y^A \frac{\partial}{\partial Y^A} + \theta^\mu \frac{\partial}{\partial \theta^\mu} + \mu \frac{\partial}{\partial \mu}$, we choose $\boldsymbol{\mathcal{F}} = \rho \boldsymbol{\mathcal{H}}^{\mathrm{T}}$, $\boldsymbol{\mathcal{G}} = \Omega_{-1} \boldsymbol{\mathcal{H}}^{\mathrm{T}}$. It then follows from Proposition A.1 that the system can be reduced to $(\widetilde{\Omega}, \Gamma(\boldsymbol{\mathcal{E}}))$. We still have to compute

$$\widetilde{\Omega} = \overset{\boldsymbol{\mathcal{E}}\boldsymbol{\mathcal{E}}}{\Omega} - \overset{\boldsymbol{\mathcal{E}}\boldsymbol{\mathcal{F}}}{\Omega} (\overset{\boldsymbol{\mathcal{G}}\boldsymbol{\mathcal{F}}}{\Omega})^{-1} \overset{\boldsymbol{\mathcal{G}}\boldsymbol{\mathcal{E}}}{\Omega}, \tag{B.1}$$

where in our case $\overset{\boldsymbol{\mathcal{E}}\boldsymbol{\mathcal{E}}}{\Omega} = 0$. For this purpose, we introduce the additional degree $\theta^\mu \frac{\partial}{\partial \theta^\mu} + \mu \frac{\partial}{\partial \mu}$, which we denote by a superscript and according to which Ω_0 decomposes as $\Omega^0_0 = c_0 \Box$, while $\Omega^1_0 = \theta^\mu (\frac{\partial}{\partial x^\mu} - \omega_{\mu}{}^{A}{}_{B} Y^B \frac{\partial}{\partial Y^A}) - \mu Y^A \frac{\partial}{\partial Y^A} + 2c_0 \mu \frac{\partial}{\partial c_0}$. As in subsection 4.2 of [25], we then get, for $\boldsymbol{\phi}^{\boldsymbol{\mathcal{E}}} \in \Gamma(\boldsymbol{\mathcal{E}})$,

$$\begin{aligned}
\widetilde{\Omega} \boldsymbol{\phi}^{\boldsymbol{\mathcal{E}}} &= \overset{\boldsymbol{\mathcal{E}}\boldsymbol{\mathcal{F}}}{\Omega^0_0} \sum_{n=1} (-1)^n \Big(N^{-1} \rho \Omega^1_0 \Big)^n \boldsymbol{\phi}^{\boldsymbol{\mathcal{E}}} \\
&= \overset{\boldsymbol{\mathcal{E}}\boldsymbol{\mathcal{F}}}{\Omega^0_0} \sum_{n=1} \frac{1}{n!} \Big(y^a e^\mu_a \Big(\frac{\partial}{\partial x^\mu} - \omega_{\mu}{}^{A}{}_{B} Y^B \frac{\partial}{\partial Y^A} \Big) - z \Big(Y^A \frac{\partial}{\partial Y^A} + 2c_0 \frac{\partial}{\partial c_0} \Big) \Big)^n \boldsymbol{\phi}^{\boldsymbol{\mathcal{E}}}, \\
&= c_0 \Box \frac{1}{2} \Big(y^a e^\mu_a \Big(\frac{\partial}{\partial x^\mu} - \omega_{\mu}{}^{A}{}_{B} Y^B \frac{\partial}{\partial Y^A} \Big) - z Y^A \frac{\partial}{\partial Y^A} \Big) \Big(Y^b e^\nu_b \frac{\partial \boldsymbol{\phi}^{\boldsymbol{\mathcal{E}}}}{\partial x^\nu} \Big) \\
&= c_0 e^{\mu a} \Big(\delta^b_a \frac{\partial}{\partial x^\mu} - \omega_{\mu}{}^{b}{}_{a} \Big) \Big(e^\nu_b \frac{\partial \boldsymbol{\phi}^{\boldsymbol{\mathcal{E}}}}{\partial x^\nu} \Big) \\
&= c_0 \Box_{\mathrm{AdS}} \boldsymbol{\phi}^{\boldsymbol{\mathcal{E}}}.
\end{aligned} \tag{B.2}$$

As an additional remark let us note that an alternative way to arrive at the statement is to take as a degree minus the homogeneity in μ and θ^ν so that $\Omega_{-1} = \nabla + \sigma + \mu(h - c_0 \frac{\partial}{\partial c_0})$. By expanding in Y^A one finds that cohomology of Ω_{-1} can be identified with $\Gamma(\boldsymbol{\mathcal{E}})$, i.e., θ^μ, μ, Y^A-independent sections. Using then a generalization of A.1 discussed in appendix A, one can reduce the system to $(\widetilde{\Omega}, \Gamma(\boldsymbol{\mathcal{E}}))$. Because the cohomology is concentrated in zeroth degree, the reduced BRST operator $\widetilde{\Omega}$ is just $\Omega_0 = c_0\Box$ understood as acting in the cohomology and coincides with (B.2).

B.2 Unfolded form

Because both Ω_{-1} and the chosen degree (minus homogeneity in μ and c_0) does not depend on the choice of frame, we are free to use the frame where $V^A = l\delta^A_{(d)}$. The operator Ω_{-1} then has the form $\Omega_{-1} = c_0\Box + \mu h - 2\mu c_0 \frac{\partial}{\partial c_0}$ where

$$h = -y^a \frac{\partial}{\partial y^a} - (z+1)\frac{\partial}{\partial z}\,, \qquad \Box = \frac{\partial}{\partial y^a}\frac{\partial}{\partial y_a} - \frac{1}{l^2}\frac{\partial}{\partial z}\frac{\partial}{\partial z}\,. \tag{B.3}$$

One first evaluates the cohomology of $c_0\Box$. Because any element is in the image of $\Box$, the cohomology is given by c_0-independent elements annihilated by $\Box$. By expanding in c_0, one concludes that the cohomology of Ω_{-1} is given by cohomology of μh in the space of c_0-independent elements annihilated by $\Box$. Using the homogeneity degree in z one then concludes that the latter cohomology is determined by the cohomology of $\mu\frac{\partial}{\partial z}$ in $\ker\Box$. We will now show that this cohomology is given by μ, z-independent elements in $\ker\Box$. Indeed, this cohomology is isomorphic to the cohomology of $c_0\Box + \mu\frac{\partial}{\partial z}$ in the space of all μ, z, c_0-dependent elements: by expanding in c_0 one observes again that any cocycle can be assumed c_0-independent and thus belonging to $\ker\Box$, while expanding in μ, one observes that any cocycle can be assumed μ, z-independent. Finally, the cohomology of $c_0\Box$ in the cohomology of $\mu\frac{\partial}{\partial z}$ is given by c_0, z, μ-independent elements from $\ker\Box$.

In degree 0, the cohomology is $\mathcal{E} = \operatorname{Ker} h \cap \operatorname{Ker}\Box$ and, by the above reasoning, this space is isomorphic to the kernel of $\Box$ in the space of power series depending on $y^a = Y^a$ alone. We will now show how to uniquely lift such an element to a z-dependent element that belongs to $\mathcal{E}$. Any element in $\operatorname{Ker} h$ is uniquely determined by its z independent part: if this part vanishes, the element vanishes and, furthermore, any element $\phi_0 \in \ker\Box$ that does not depend on z can be completed to a unique solution of $h\phi = 0$ and $\Box\phi = 0$. In fact ϕ can be constructed order by order in z using the fact that any element in $\ker\Box$ is in the image of $\frac{\partial}{\partial z}$ in $\ker\Box$. This is just a reformulation of the fact that $\mu\frac{\partial}{\partial z}$ does not have cohomology in non-vanishing degree in μ, z.

The explicit expansion of ϕ in terms of z and $y^a y_a$ has the following form:

$$\phi = \frac{1}{(1+z)^n}\left(\phi_0 + (y^a y_a)\frac{n(n+1)}{2l^2(d+2n)}\phi_0 + \cdots\right), \tag{B.4}$$

where $n = y^a\frac{\partial}{\partial y^a}$, the ratio is understood as a formal power series, and $\ldots$ denote terms of the form $(y^a y_a)^k\phi_0,\ k \geqslant 2$. The coefficients in front of the terms $(y^a y_a)^k\phi_0$ are uniquely determined by the requirement $\Box\phi = 0$.

C. Higher spin gauge fields on AdS: details of reductions

C.1 Standard description

To get the intermediate reduction to tensor fields taking values in the embedding space, one chooses as a grading $y^a\frac{\partial}{\partial y^a}+2c_0\frac{\partial}{\partial c_0}-b^\dagger\frac{\partial}{\partial b^\dagger}+c^\dagger\frac{\partial}{\partial c^\dagger}$. The BRST operator (4.14) decomposes as

$$\begin{aligned}
\Omega_{-1} &= -\theta^\mu(1+z)e^a_\mu\frac{\partial}{\partial y^a}\,,\\
\Omega_0 &= \theta^\mu\frac{\partial}{\partial x^\mu}-\theta^\mu(\omega_{\mu}{}^b{}_c y^c\frac{\partial}{\partial y^b}+\omega_{\mu}{}^B{}_C a^{\dagger C}\frac{\partial}{\partial a^{\dagger B}})+c_0\frac{\partial}{\partial y_a}\frac{\partial}{\partial y^a}+a^{\dagger a}\frac{\partial}{\partial y^a}\frac{\partial}{\partial b^\dagger}+\\
&\quad +c^\dagger\frac{\partial}{\partial a^\dagger_a}\frac{\partial}{\partial y^a}+\xi T+\mu(-\frac{\partial}{\partial z}-Y^A\frac{\partial}{\partial Y^A}+a^{\dagger A}\frac{\partial}{\partial a^{\dagger A}})+\nu(\frac{\partial}{\partial w}+z\frac{\partial}{\partial w})-\\
&\quad -c^\dagger\frac{\partial}{\partial b^\dagger}\frac{\partial}{\partial c_0}-2\xi\frac{\partial}{\partial b^\dagger}\frac{\partial}{\partial c^\dagger}-2\mu c_0\frac{\partial}{\partial c_0}-2\mu\frac{\partial}{\partial b^\dagger}b^\dagger+2\mu\xi\frac{\partial}{\partial \xi}+2\mu\nu\frac{\partial}{\partial \nu}\,,\\
\Omega_1 &= -\theta^\mu\frac{e^a_\mu}{l^2}y_a\frac{\partial}{\partial z}+w\frac{\partial}{\partial z}\frac{\partial}{\partial b^\dagger}-\frac{c^\dagger}{l^2}\frac{\partial}{\partial w}\frac{\partial}{\partial z}+\nu y^a\frac{\partial}{\partial a^{\dagger a}}+2\nu c_0\frac{\partial}{\partial c^\dagger}+\nu c^\dagger\frac{\partial}{\partial \xi}+\nu\frac{\partial}{\partial b^\dagger}\frac{\partial}{\partial \mu}\,,\\
\Omega_2 &= -\frac{1}{l^2}c_0\frac{\partial}{\partial z}\frac{\partial}{\partial z}\,.
\end{aligned}\tag{C.1}$$

In this case, $\rho=-\frac{y^a}{1+z}e^\mu_a\frac{\partial}{\partial\theta^\mu}$ and $N=y^a\frac{\partial}{\partial y^a}+\theta^\mu\frac{\partial}{\partial\theta^\mu}$, which determines the decomposition $\boldsymbol{\mathcal{H}}^{\mathbf{T}}=\boldsymbol{\mathcal{E}}\oplus\boldsymbol{\mathcal{G}}\oplus\boldsymbol{\mathcal{F}}$ with $\Gamma(\boldsymbol{\mathcal{E}})\cong H(\Omega_{-1},\Gamma(\boldsymbol{\mathcal{H}}^{\mathbf{T}}))$ in the same way as in B.1. The additional degree is $\theta^\mu\frac{\partial}{\partial\theta^\mu}$, so that

$$\begin{aligned}
\Omega^0_0 &= c_0\frac{\partial}{\partial y_a}\frac{\partial}{\partial y^a}+a^{\dagger a}\frac{\partial}{\partial y^a}\frac{\partial}{\partial b^\dagger}+\\
&\quad +c^\dagger\frac{\partial}{\partial a^\dagger_a}\frac{\partial}{\partial y^a}+\xi T+\mu(-\frac{\partial}{\partial z}-Y^A\frac{\partial}{\partial Y^A}+a^{\dagger A}\frac{\partial}{\partial a^{\dagger A}})+\nu(\frac{\partial}{\partial w}+z\frac{\partial}{\partial w})-\\
&\quad -c^\dagger\frac{\partial}{\partial b^\dagger}\frac{\partial}{\partial c_0}-2\xi\frac{\partial}{\partial b^\dagger}\frac{\partial}{\partial c^\dagger}-2\mu c_0\frac{\partial}{\partial c_0}-2\mu\frac{\partial}{\partial b^\dagger}b^\dagger+2\mu\xi\frac{\partial}{\partial \xi}+2\mu\nu\frac{\partial}{\partial \nu}\,,\\
\Omega^0_1 &= w\frac{\partial}{\partial z}\frac{\partial}{\partial b^\dagger}-\frac{c^\dagger}{l^2}\frac{\partial}{\partial w}\frac{\partial}{\partial z}+\nu y^a\frac{\partial}{\partial a^{\dagger a}}+2\nu c_0\frac{\partial}{\partial c^\dagger}+\nu c^\dagger\frac{\partial}{\partial \xi}+\nu\frac{\partial}{\partial b^\dagger}\frac{\partial}{\partial \mu}\,,\\
\Omega^0_2 &= -\frac{1}{l^2}c_0\frac{\partial}{\partial z}\frac{\partial}{\partial z}\,,\\
\Omega^1_0 &= \theta^\mu\frac{\partial}{\partial x^\mu}-\theta^\mu(\omega_{\mu}{}^b{}_c y^c\frac{\partial}{\partial y^b}+\omega_{\mu}{}^B{}_C a^{\dagger C}\frac{\partial}{\partial a^{\dagger B}})\,,\\
\Omega^1_1 &= -\theta^\mu\frac{e^a_\mu}{l^2}y_a\frac{\partial}{\partial z}\,.
\end{aligned}\tag{C.2}$$

We now have

$$\begin{aligned}
\overset{\boldsymbol{\mathcal{E}}\boldsymbol{\mathcal{E}}}{\Omega} &= -\frac{1}{l^2}c_0\frac{\partial}{\partial z}\frac{\partial}{\partial z}+w\frac{\partial}{\partial z}\frac{\partial}{\partial b^\dagger}-\frac{c^\dagger}{l^2}\frac{\partial}{\partial w}\frac{\partial}{\partial z}+\xi T-\\
&\quad -\mu(\frac{\partial}{\partial z}+z\frac{\partial}{\partial z}-a^{\dagger A}\frac{\partial}{\partial a^{\dagger A}})+\nu(\frac{\partial}{\partial w}+z\frac{\partial}{\partial w})-c^\dagger\frac{\partial}{\partial b^\dagger}\frac{\partial}{\partial c_0}+\nu c^\dagger\frac{\partial}{\partial \xi}+\nu\frac{\partial}{\partial b^\dagger}\frac{\partial}{\partial \mu}-\\
&\qquad -2\mu c_0\frac{\partial}{\partial c_0}-2\mu\frac{\partial}{\partial b^\dagger}b^\dagger+2\mu\xi\frac{\partial}{\partial \xi}+2\mu\nu\frac{\partial}{\partial \nu}-2\xi\frac{\partial}{\partial b^\dagger}\frac{\partial}{\partial c^\dagger}+2\nu c_0\frac{\partial}{\partial c^\dagger}\,,
\end{aligned}\tag{C.3}$$

JHEP08(2006)013

while

$$
\begin{aligned}
-\overset{\mathcal{EF}}{\Omega}(\overset{\mathcal{GF}}{\Omega})^{-1}\overset{\mathcal{GE}}{\Omega}\phi^{\mathcal{E}} &= \overset{\mathcal{EF}}{\Omega^0}\sum_{n=1}(-1)^n\big(N^{-1}\rho(\Omega_0^1+\Omega_1^1)\big)^n\phi^{\mathcal{E}} = \\
&= \overset{\mathcal{EF}}{\Omega^0}\sum_{n=1}\frac{1}{n!}\Big(\frac{y^a}{1+z}\big(\partial_a-\omega_{a\,c}^{\,b}y^c\frac{\partial}{\partial y^b}-\omega_{a\,C}^{\,B}a^{\dagger C}\frac{\partial}{\partial a^{\dagger B}}-\frac{y_a}{2l^2}\frac{\partial}{\partial z}\big)\Big)^n\phi^{\mathcal{E}} = \\
&= \Big[\frac{1}{1+z}(a^{\dagger a}\mathcal{D}_a\frac{\partial}{\partial b^\dagger}+c^\dagger\frac{\partial}{\partial a_a^\dagger}\mathcal{D}_a)+ \\
&\quad + c_0[\big(\frac{1}{1+z}\big)^2\eta^{ac}(\delta_c^b\mathcal{D}_a-\omega_{a\,c}^{\,b})\mathcal{D}_b-\frac{1}{1+z}\frac{d}{l^2}\frac{\partial}{\partial z}]\Big]\phi^{\mathcal{E}}\,, \qquad (C.4)
\end{aligned}
$$

where $\partial_a=e_\mu^a\frac{\partial}{\partial x^\mu}$, $\omega_{a\,C}^{\,B}=e_a^\mu\omega_{\mu\,C}^{\,B}$ and $\mathcal{D}_a=\partial_a-\omega_{a\,C}^{\,B}a^{\dagger C}\frac{\partial}{\partial a^{\dagger B}}$. The reduced BRST operator is then given by (4.23).

To further reduce to tensor fields on AdS, we now choose as grading $z\frac{\partial}{\partial z}+2c_0\frac{\partial}{\partial c_0}-b^\dagger\frac{\partial}{\partial b^\dagger}+c^\dagger\frac{\partial}{\partial c^\dagger}-\nu\frac{\partial}{\partial\nu}$. The BRST operator (4.23) then decomposes as

$$
\begin{aligned}
\Omega_{-1} &= -\mu\frac{\partial}{\partial z}+\nu\frac{\partial}{\partial w}\,, \\
\Omega_{\geqslant 0} &= -\frac{1}{l^2}c_0\frac{\partial}{\partial z}\frac{\partial}{\partial z}+w\frac{\partial}{\partial z}\frac{\partial}{\partial b^\dagger}-\frac{c^\dagger}{l^2}\frac{\partial}{\partial w}\frac{\partial}{\partial z}+\xi T-\mu(z\frac{\partial}{\partial z}-a^{\dagger A}\frac{\partial}{\partial a^{\dagger A}})+\nu z\frac{\partial}{\partial w}+ \\
&\quad +\frac{1}{1+z}(a^{\dagger a}\mathcal{D}_a\frac{\partial}{\partial b^\dagger}+c^\dagger\frac{\partial}{\partial a_a^\dagger}\mathcal{D}_a)+c_0[\big(\frac{1}{1+z}\big)^2\eta^{ac}(\delta_c^b\mathcal{D}_a-\omega_{a\,c}^{\,b})\mathcal{D}_b-\frac{1}{1+z}\frac{d}{l^2}\frac{\partial}{\partial z}]- \\
&\quad -c^\dagger\frac{\partial}{\partial b^\dagger}\frac{\partial}{\partial c_0}+\nu c^\dagger\frac{\partial}{\partial\xi}+\nu\frac{\partial}{\partial b^\dagger}\frac{\partial}{\partial\mu}-2\mu c_0\frac{\partial}{\partial c_0}-2\mu\frac{\partial}{\partial b^\dagger}b^\dagger+ \\
&\quad +2\mu\xi\frac{\partial}{\partial\xi}+2\mu\nu\frac{\partial}{\partial\nu}-2\xi\frac{\partial}{\partial b^\dagger}\frac{\partial}{\partial c^\dagger}+2\nu c_0\frac{\partial}{\partial c^\dagger}\,. \qquad (C.5)
\end{aligned}
$$

In this case, $\rho=-z\frac{\partial}{\partial\mu}+w\frac{\partial}{\partial\nu}$, $N=z\frac{\partial}{\partial z}+w\frac{\partial}{\partial w}+\mu\frac{\partial}{\partial\mu}+\nu\frac{\partial}{\partial\nu}$. The additional degree is $\mu\frac{\partial}{\partial\mu}+\nu\frac{\partial}{\partial\nu}$ so that

$$
\begin{aligned}
\Omega_{\geqslant 0}^0 &= -\frac{1}{l^2}c_0\frac{\partial}{\partial z}\frac{\partial}{\partial z}+w\frac{\partial}{\partial z}\frac{\partial}{\partial b^\dagger}-\frac{c^\dagger}{l^2}\frac{\partial}{\partial w}\frac{\partial}{\partial z}+\xi T+\frac{1}{1+z}(a^{\dagger a}\mathcal{D}_a\frac{\partial}{\partial b^\dagger}+c^\dagger\frac{\partial}{\partial a_a^\dagger}\mathcal{D}_a) \\
&\quad +c_0[\big(\frac{1}{1+z}\big)^2\eta^{ac}(\delta_c^b\mathcal{D}_a-\omega_{a\,c}^{\,b})\mathcal{D}_b-\frac{1}{1+z}\frac{d}{l^2}\frac{\partial}{\partial z}]- \\
&\quad -c^\dagger\frac{\partial}{\partial b^\dagger}\frac{\partial}{\partial c_0}+\nu\frac{\partial}{\partial b^\dagger}\frac{\partial}{\partial\mu}-2\xi\frac{\partial}{\partial b^\dagger}\frac{\partial}{\partial c^\dagger}\,, \\
\Omega_{\geqslant 0}^1 &= \Omega_0^1=-\mu(z\frac{\partial}{\partial z}-a^{\dagger A}\frac{\partial}{\partial a^{\dagger A}})+\nu z\frac{\partial}{\partial w}++\nu c^\dagger\frac{\partial}{\partial\xi}-2\mu c_0\frac{\partial}{\partial c_0} \\
&\quad -2\mu\frac{\partial}{\partial b^\dagger}b^\dagger+2\mu\xi\frac{\partial}{\partial\xi}+2\mu\nu\frac{\partial}{\partial\nu}+2\nu c_0\frac{\partial}{\partial c^\dagger}\,. \qquad (C.6)
\end{aligned}
$$

We now have

$$
\begin{aligned}
\overset{\mathcal{EE}}{\Omega} &= \xi\frac{\partial}{\partial a^{\dagger a}}\frac{\partial}{\partial a_a^\dagger}+(c^\dagger\frac{\partial}{\partial a_a^\dagger}D_a+a^{\dagger a}D_a\frac{\partial}{\partial b^\dagger})+ \\
&\quad +c_0[\eta^{ac}(\delta_c^bD_a-\omega_{a\,c}^{\,b})D_b+\frac{1}{l^2}a^{\dagger a}\frac{\partial}{\partial a_a^\dagger}]-c^\dagger\frac{\partial}{\partial b^\dagger}\frac{\partial}{\partial c_0}-2\xi\frac{\partial}{\partial b^\dagger}\frac{\partial}{\partial c^\dagger}\,, \qquad (C.7)
\end{aligned}
$$

where

$$D_a = \partial_a - \omega_{a\,c}^{b} a^{\dagger c} \frac{\partial}{\partial a^{\dagger b}} \,, \quad [D_a, D_b] = (\omega_{ab}^{c} - \omega_{ba}^{c}) D_c - \frac{1}{l^2} (a_a^{\dagger} \frac{\partial}{\partial a^{\dagger b}} - a_b^{\dagger} \frac{\partial}{\partial a^{\dagger a}}) \,, \tag{C.8}$$

while

$$\begin{aligned}
-\overset{\boldsymbol{\mathcal{EF}}}{\Omega} (\overset{\boldsymbol{\mathcal{GF}}}{\Omega})^{-1} \overset{\boldsymbol{\mathcal{GE}}}{\Omega} \boldsymbol{\phi}^{\boldsymbol{\mathcal{E}}} &= \overset{\boldsymbol{\mathcal{EF}}}{\Omega^0} \sum_{n=1} (-1)^n \big(N^{-1} \rho \Omega^1_{\geqslant 0} \big)^n \boldsymbol{\phi}^{\boldsymbol{\mathcal{E}}} = \\
&= \overset{\boldsymbol{\mathcal{EF}}}{\Omega^0} \Big(zL + wK + \frac{1}{2}(zL + wK)^2 - \frac{1}{2} z^2 L + \dots \Big) \boldsymbol{\phi}^{\boldsymbol{\mathcal{E}}} = \\
&= \Big(\frac{c_0}{l^2} \big(L(1 - d - L) - 2K a^{\dagger a} D_a + \frac{K^2 a^{\dagger a} a_a^{\dagger}}{l^2} \big) - \\
&\quad - \frac{c^{\dagger}}{l^2} \big(\frac{1}{2} \{L, K\} + K(d + a^{\dagger a} \frac{\partial}{\partial a^{\dagger a}}) \big) - \frac{K a^{\dagger a} a_a^{\dagger}}{l^2} \frac{\partial}{\partial b^{\dagger}} - \frac{\xi}{l^2} K^2 \Big) \boldsymbol{\phi}^{\boldsymbol{\mathcal{E}}} \,,
\end{aligned} \tag{C.9}$$

where $\dots$ mean irrelevant terms of order at least 3 in w, z, while $L = a^{\dagger a} \frac{\partial}{\partial a^{\dagger a}} - 2c_0 \frac{\partial}{\partial c_0} - 2 \frac{\partial}{\partial b^{\dagger}} b^{\dagger} + 2\xi \frac{\partial}{\partial \xi}$ and $K = -c^{\dagger} \frac{\partial}{\partial \xi} - 2c_0 \frac{\partial}{\partial c^{\dagger}}$. By noting that $\frac{1}{2}\{K, L\} = (L+1)K$, $K^2 = 2c_0(1 - 2N_{c^{\dagger}}) \frac{\partial}{\partial \xi}$ and using (C.7) and (C.9) in the definition (2.3), we get the result (4.30) of section 4.3.

C.2 Unfolded form

Proof of Proposition 4.2 To prove that $H^n(\widetilde{\Omega}_{-1}, \widetilde{\mathcal{E}}) = 0$ for $n \neq 0$, let us suppose that $V^A = l\delta^A_{(d)}$ and introduce the following notations: $z^1 = z = Y^{(d)}/l, z^2 = w = a^{\dagger (d)}/l$ and $\mu^1 = \mu, \mu^2 = \nu$.

By expanding in the homogeneity in Y and $a^{\dagger}$, one finds that the cohomology of $\widetilde{\Omega}_{-1}$ is controlled by the cohomology of $\delta = \mu^{\alpha} \frac{\partial}{\partial z^{\alpha}}$. As a preliminary result, we need to show that the cohomology of δ, which is trivial in the space of formal power series in Y with coefficients that are polynomials in $a^{\dagger A}, \mu^{\alpha}$ and the other ghost variables, is also trivial in the space of traceless elements:

Lemma C.1. *The cohomology of δ in $\widetilde{\mathcal{E}}$, the space of completely traceless elements, is given by μ^{α}, z^{α}-independent traceless elements.*

Proof. The cohomology of δ in $\widetilde{\mathcal{E}}$ can be represented as that of $\Omega_{\text{trace}} + \delta$ in the space $\mathcal{H}^{\mathrm{T}}$ where Ω_{trace} is given by (4.34). Indeed, taking as a degree minus the homogeneity in ghosts $c_0, c^{\dagger}, \xi$ and using the fact that the cohomology of Ω_{trace} is concentrated in zeroth degree one concludes that the cohomology of $\Omega_{\text{trace}} + \delta$ is given by cohomology of δ in $\widetilde{\mathcal{E}}$.

On the other hand, taking as a degree minus the homogeneity in μ^{α} one finds that $\deg \delta = -1$, $\deg \Omega_{\text{trace}} = 0$. The cohomology of δ is concentrated in zeroth degree and is given by z^{α}, μ^{α}-independent elements. The cohomology of the entire operator is then given by that of Ω_{trace} reduced to the subspace of z^{α}, μ^{α}-independent elements. The reduced differential is given by

$$\begin{gathered}
\Omega^0_{\text{trace}} = c_0 \Box_0 + c^{\dagger} T_0 + \xi S_0 \,, \\
\Box_0 = \frac{\partial^2}{\partial y^a \partial y_a} \,, \qquad T_0 = \frac{\partial^2}{\partial a^{\dagger a} \partial a_a^{\dagger}} \,, \qquad S_0 = \frac{\partial^2}{\partial y^a \partial a_a^{\dagger}} \,.
\end{gathered} \tag{C.10}$$

According to the results of [25], the cohomology of Ω^0_{trace} is given by $c_0, c^\dagger, \xi$-independent traceless elements provided $d \geqslant 3$. $\square$

Analogous arguments show that the cohomology of $\delta^1 = \mu^1 \frac{\partial}{\partial z^1}$ (respectively $\delta^2 = \mu^2 \frac{\partial}{\partial z^2}$) in the space of traceless elements is given by μ^1, z^1- (respectively μ^2, z^2) independent traceless elements. Let $\widetilde{\mathcal{E}}_0 \subset \widetilde{\mathcal{E}}$ be the subspace of μ^α, z^α-independent elements. One then has the following:

Proposition C.2. *For any $\phi \in \widetilde{\mathcal{E}}_0$ satisfying $\bar{S}^\dagger \phi = 0$ and an arbitrary number m, there exists a unique z^α-dependent and μ^α-independent $A \in \widetilde{\mathcal{E}}$ such that*

$$\phi = (h - m)A\,, \quad \bar{S}^\dagger A = 0\,, \quad \mathcal{P}(A|_{z^\alpha=0}) = 0\,, \tag{C.11}$$

where $\mathcal{P}$ is the projector to the subspace of totally traceless elements in the space of z^α independent elements.

Proof. Multiplying equations (C.11) respectively by μ^1 and μ^2, one gets

$$\delta A = \Delta A - \mu^1 \phi\,, \qquad \Delta = \mu^1 (h - m + \frac{\partial}{\partial z^1}) - \mu^2 (\bar{S}^\dagger - \frac{\partial}{\partial z^2})\,, \quad \delta = \mu^\alpha \frac{\partial}{\partial z^\alpha}\,. \tag{C.12}$$

Note that Δ is homogeneous of degree 0 in the total degree that counts Y's and $a^\dagger$'s. By expanding according to this total degree, one gets the equations

$$\delta A_{n+1} = \Delta A_n - \mu^1 \phi_n\,, \tag{C.13}$$

which has a unique μ^α-independent solution satisfying the condition $\mathcal{P}(A_n|_{z^\alpha=0}) = 0$ for all n. This follows from the fact that the cohomology of δ is given by z^α and μ^α independent elements while the consistency holds due to the following identities

$$[\delta, \Delta] = -2\mu^1 \delta\,, \qquad \frac{1}{2}[\Delta, \Delta] = -2\mu^1 \Delta\,, \tag{C.14}$$

and $\bar{S}^\dagger \phi = 0$. In order to see that the solution is unique one notes that the arbitrariness in A_{n+1} is given by z^α-independent terms which are traceless. By requiring the traceless part of $A|_{z^\alpha=0}$ to vanish one thus fixes the ambiguity. Note that in general A is a formal power series in z^1 even if ϕ is polynomial in all the variables. $\square$

Similar arguments show that any element from $\widetilde{\mathcal{E}}_0$ is in the image of $\bar{S}^\dagger$. One then has all the ingredients needed in the proof of Proposition 4.2.

Proof of Proposition 4.5 First, one observes that the second term in (4.50) vanishes when acting on a $\mathcal{E}_1$-valued section and therefore one gets

$$\Omega^{\text{unf}} \boldsymbol{\chi} = (\nabla + \sigma)\boldsymbol{\chi}\,, \qquad \boldsymbol{\chi} \in \Gamma(\boldsymbol{\mathcal{E}_1})\,. \tag{C.15}$$

Note that the projection here is not needed because $D = \nabla + \sigma$ preserves $\Gamma(\boldsymbol{\mathcal{E}_1})$.

To compute $\Omega^{\text{unf}} \boldsymbol{\phi}$ for $\boldsymbol{\phi} \in \Gamma(\boldsymbol{\mathcal{E}}_0)$ it is convenient to choose the frame where $V^A = l\delta^A_{(d)}$. One is allowed to use a special frame because the statement is frame-independent. As usually we also use $lz = lz^1 = Y^{(d)}$, $lw = lz^2 = a^{\dagger(d)}$. For a $\mathcal{E}_0$-valued section again $\overset{\boldsymbol{\mathcal{E}\mathcal{E}}}{D}\boldsymbol{\phi} = D\boldsymbol{\phi}$ but the second term in (4.50) can be non-vanishing. To compute it explicitly, one first observes that it can be non-vanishing only on elements whose traceless part at $z^\alpha = 0$ is annihilated by h_0, i.e., is described by a rectangular Young tableaux. This follows from counting homogeneity degree in $y^a, a^{\dagger a}$. Let $\boldsymbol{\phi} \in \Gamma(\boldsymbol{\mathcal{E}}_0)$ be such that the traceless part $\boldsymbol{\phi}_0$ of $\boldsymbol{\phi}|_{z^\alpha=0}$ satisfies $h\boldsymbol{\phi}_0 = h_0\boldsymbol{\phi}_0 = 0$. One then observes that

$$\mathcal{P}_{\mathcal{G}} D\boldsymbol{\phi} = -S^\dagger \mathcal{K}_0(\bar{\sigma}\boldsymbol{\phi}_0)\,, \qquad \bar{\sigma} = -\theta^\mu e^A_\mu \frac{\partial}{\partial a^{\dagger A}}\,, \tag{C.16}$$

where $\boldsymbol{X} = \mathcal{K}_0(\bar{\sigma}\boldsymbol{\phi}_0)$ is a unique solution $\boldsymbol{X} \in \Gamma(\widehat{\boldsymbol{\mathcal{E}}})$ to the equation $h\boldsymbol{X} = \bar{S}^\dagger\boldsymbol{X} = 0$ satisfying $\mathcal{P}(\boldsymbol{X}|_{z^\alpha=0}) = \bar{\sigma}\boldsymbol{\phi}_0$ (see Proposition 4.4). Indeed, to see this it is enough to show that $D\boldsymbol{\phi} + S^\dagger\boldsymbol{X} \in \Gamma(\boldsymbol{\mathcal{E}_0})$ which is in turn equivalent to $\frac{\partial}{\partial w}(D\boldsymbol{\phi} + S^\dagger\boldsymbol{X}) = 0$. One then uses $\frac{\partial}{\partial w}\boldsymbol{X} = \frac{\partial}{\partial w}\boldsymbol{\phi} = 0$, $[\frac{\partial}{\partial w}, D] = \bar{\sigma}$, $[\frac{\partial}{\partial w}, S^\dagger] = \frac{\partial}{\partial z}$ and finds that both $\frac{\partial}{\partial z}\boldsymbol{X}$ and $-\bar{\sigma}\boldsymbol{\phi}$ are annihilated by $h-1$ and $\bar{S}^\dagger$ and satisfy the same boundary condition at $z=0$. It then follows from the Proposition 4.4 that they do coincide.

By counting degree in y^b and $a^{\dagger b}$ one finds that $\mathcal{P}_{\mathcal{E}_1} b^\dagger \mathcal{K}_0(\bar{\sigma}\boldsymbol{\phi}_0) = 0$ because $b^\dagger\bar{\sigma}\boldsymbol{\phi}_0$ does not belong to $\bar{\mathcal{E}}_1$. Thus

$$\rho \overset{\mathcal{G}\mathcal{E}_0}{D}\boldsymbol{\phi} = -b^\dagger \mathcal{K}_0(\bar{\sigma}\boldsymbol{\phi}_0)\,. \tag{C.17}$$

Again, by counting the degree one finds that the only contribution to $\mathcal{E}_1$ from $-Db^\dagger\mathcal{K}_0(\bar{\sigma}\boldsymbol{\phi}_0)$ comes from the terms in D that lower the degree in y^a. These terms give $(z+1)\sigma$ and one finds that

$$\mathcal{P}_{\mathcal{E}_1} b^\dagger D\mathcal{K}_0(\bar{\sigma}\boldsymbol{\phi}_0) = \mathcal{K}_0(b^\dagger\sigma\bar{\sigma}\boldsymbol{\phi}_0) = b^\dagger\sigma\bar{\sigma}\boldsymbol{\phi}_0 \tag{C.18}$$

where we have used $[h,(z+1)\sigma] = 0$, $[\bar{S}^\dagger,(z+1)\sigma] = -(z+1)\bar{\sigma}$, and $\bar{\sigma}\mathcal{K}(\bar{\sigma}\boldsymbol{\phi}_0) = 0$. The last equality follows from the fact that $h_0\sigma\bar{\sigma}\boldsymbol{\phi}_0 = \bar{S}^\dagger_0\sigma\bar{\sigma}\boldsymbol{\phi}_0 = S^\dagger_0\sigma\bar{\sigma}\boldsymbol{\phi}_0 = 0$ and therefore $\mathcal{K}_0\sigma\bar{\sigma}\boldsymbol{\phi}_0 = \sigma\bar{\sigma}\boldsymbol{\phi}_0$.

There remains to show that for arbitrary $\boldsymbol{\phi} \in \Gamma(\boldsymbol{\mathcal{E}}_0)$ one has $\mathcal{P}_R\boldsymbol{\phi} = \mathcal{P}^0_R\boldsymbol{\phi}_0$ where $\mathcal{P}^0_R$ denotes the standard projector onto the subspace of elements in $\bar{\mathcal{E}}_0$ described by rectangular Young tableaux. Indeed, it follows from $\bar{S}^\dagger_0\boldsymbol{\phi}_0 = 0$ and the explicit structure (4.54) of $\boldsymbol{\phi} = \mathcal{K}_2\boldsymbol{\phi}_0$ that all monomials in $\boldsymbol{\phi} - \boldsymbol{\phi}_0$ contain more Y^A variables than $a^{\dagger A}$ ones. This allows to rewrite the contribution (C.18) in frame-independent terms

$$b^\dagger\sigma\bar{\sigma}\mathcal{P}^0_R\boldsymbol{\phi}_0 = b^\dagger\sigma\bar{\sigma}\mathcal{P}_R\boldsymbol{\phi}\,, \tag{C.19}$$

where $\boldsymbol{\phi}_0 = \mathcal{P}(\boldsymbol{\phi}|_{z^\alpha=0})$ and $\boldsymbol{\phi} \in \mathcal{E}_0$.

Proof of Proposition 4.6 First one observes that $D_{\bar{\mathcal{E}}_0}\boldsymbol{\phi}_0 = \mathcal{P}_{\bar{\mathcal{E}}_0}\mathcal{P}[(D\boldsymbol{\phi})|_{z^\alpha=0}]$ where $\boldsymbol{\phi} \in \Gamma(\boldsymbol{\mathcal{E}_0})$ satisfies $\mathcal{P}[\boldsymbol{\phi}|_{z^\alpha=0}] = \boldsymbol{\phi}_0$. To compute $D_{\bar{\mathcal{E}}_0}\boldsymbol{\phi}_0$ explicitly one then observes that the first two terms in (4.56) obviously commute with putting z^α to zero as well as with the projection to the subspace of elements annihilated by $\bar{S}^\dagger_0$. From the rest of the terms the third and the fourth ones give some additional contributions.

In order to find $\mathcal{P}_{\bar{\mathcal{E}}_0}\mathcal{P}[(\sigma\boldsymbol{\phi})|_{z^\alpha=0}]$ one notices that it follows from $(h-2)\boldsymbol{\phi} = 0$ that the expansion of $\boldsymbol{\phi}$ in z has the form

$$\boldsymbol{\phi} = \frac{1}{(z+1)^{n-s+2}}\boldsymbol{\phi}|_{z=0}\,, \tag{C.20}$$

where $n = y^a \frac{\partial}{\partial y^a}$ and $s = a^{\dagger a} \frac{\partial}{\partial a^{\dagger a}}$ are the operators counting the degree of homogeneity in y^a and $a^{\dagger a}$. On the other hand, taking a $\Box$-traceless part of $\Box \boldsymbol{\phi} = 0$ at $z = 0$ and using $\Box_z = -\frac{1}{l^2} \frac{\partial}{\partial z} \frac{\partial}{\partial z}$, one finds

$$\boldsymbol{\phi}|_{z=0} = \boldsymbol{\phi}_0 + \frac{(n-s)(n-s+1)}{2l^2(d+2n-4)} (y^a y_a) \boldsymbol{\phi}_0 + \cdots \tag{C.21}$$

where $\ldots$ denote terms proportional to higher powers in $y^a y_a$. Computing the projector explicitly one then finds

$$\begin{aligned} \mathcal{P}_{\bar{\mathcal{E}}_0}(\sigma \boldsymbol{\phi}|_{z^\alpha=0}) = \sigma \boldsymbol{\phi}_0 + \frac{1}{n-s+1} S_0^\dagger \bar{\sigma} \boldsymbol{\phi}_0 - \\ - \frac{(n-s+1)(n-s+2)}{l^2(d+2n-2)} \mathcal{P}\left[(e^a y_a) \boldsymbol{\phi}_0\right] . \end{aligned} \tag{C.22}$$

It follows from the expansions (C.20) and (C.21) that

$$\mathcal{P}[(\frac{\partial}{\partial z} \boldsymbol{\phi})_{z^\alpha=0}] = -(n-s+2) \boldsymbol{\phi}_0 \,, \tag{C.23}$$

which in turn implies

$$\mathcal{P}_{\bar{\mathcal{E}}_0} \mathcal{P}\left[(-\frac{1}{l^2} e^a y_a \frac{\partial}{\partial z} \boldsymbol{\phi})|_{z^\alpha=0}\right] = \frac{n-s+1}{l^2} \mathcal{P}\left[e^a y_a \boldsymbol{\phi}_0\right] . \tag{C.24}$$

Note that here the projection $\mathcal{P}_{\bar{\mathcal{E}}_0}$ is omitted because $\bar{S}_0^\dagger$ commutes with $e^a y_a$ so that the result belongs to $\bar{\mathcal{E}}_0$ automatically. Summing up all contribution one arrives at (4.59).

For $D_{\bar{\mathcal{E}}_1}$ one finds $D_{\bar{\mathcal{E}}_1} \boldsymbol{\chi}_0 = \mathcal{P}_{\bar{\mathcal{E}}_1} \mathcal{P}[(D\boldsymbol{\chi})|_{z^\alpha=0}]$ where $\boldsymbol{\chi}$ is uniquely determined by $b^\dagger \boldsymbol{\chi} \in \Gamma(\boldsymbol{\mathcal{E}_1})$ and $\mathcal{P}[\boldsymbol{\chi}|_{z^\alpha=0}] = \boldsymbol{\chi}_0$. In this case one observes that only the first five terms in (4.56) do contribute to $\mathcal{P}_{\bar{\mathcal{E}}_1} \mathcal{P}[(D\boldsymbol{\chi})|_{z^\alpha=0}]$. Finding the explicit expression is a bit more difficult in this case. The relevant terms in the expansion of $\boldsymbol{\chi}$ in terms of z, w and traces read as:

$$\begin{aligned} \boldsymbol{\chi} = \boldsymbol{\chi}_0 + z(s-n)\boldsymbol{\chi}_0 - w\bar{S}_0^\dagger \boldsymbol{\chi}_0 + \\ + \frac{1}{2}(s-n)(s-n+1) zz \boldsymbol{\chi}_0 - (s-n+1) zw \bar{S}_0^\dagger \boldsymbol{\chi}_0 + \frac{1}{2} ww \bar{S}_0^\dagger \bar{S}_0^\dagger \boldsymbol{\chi}_0 + \cdots + \\ + \alpha(y^a y_a)\boldsymbol{\chi}_0 + \beta(y^a a_a^\dagger) \bar{S}_0^\dagger \boldsymbol{\chi}_0 + \gamma(a^{\dagger a} a_a^\dagger) \bar{S}_0^\dagger \bar{S}_0^\dagger \boldsymbol{\chi}_0 + \cdots \end{aligned} \tag{C.25}$$

where α, β, γ are coefficients depending on n, s. It is useful to assume for the moment that $\boldsymbol{\chi}_0$ is homogeneous so that $n\boldsymbol{\chi}_0 = \bar{n}$ and $s\boldsymbol{\chi}_0 = \bar{s}$. The tracelessness condition then implies:

$$\begin{aligned} -\frac{1}{l^2}(\bar{s}-\bar{n})(\bar{s}-\bar{n}-1) + 2\alpha(d+2\bar{n}) + 2\beta(\bar{s}-\bar{n}) = 0 \,, \\ \frac{1}{l^2}(\bar{s}-\bar{n}-1) + 2\alpha + \beta(d+\bar{n}+\bar{s}) + 4\gamma(\bar{s}-\bar{n}-1) = 0 \,, \\ -\frac{1}{l^2} + 2\beta + 2\gamma(d+2\bar{s}-4) = 0 \,. \end{aligned} \tag{C.26}$$

which determines the coefficients to be

$$\alpha = \frac{(\bar{s}-\bar{n})(\bar{s}-\bar{n}-1)}{2l^2(d+2\bar{n}-2)} \,, \quad \beta = -\frac{\bar{s}-\bar{n}-1}{l^2(d+2\bar{n}-2)} \,, \quad \gamma = \frac{1}{2l^2(d+2\bar{n}-2)} . \tag{C.27}$$

Now one can explicitly compute all contributions:

$$\begin{aligned}\mathcal{P}_{\bar{\mathcal{E}}_1}\mathcal{P}\left[(\sigma\boldsymbol{\chi})|_{z^\alpha=0}\right] = \sigma\boldsymbol{\chi}_0 - 2\alpha\mathcal{P}_{\bar{\mathcal{E}}_1}\mathcal{P}[e^a y_a\boldsymbol{\chi}_0] - \beta\mathcal{P}_{\bar{\mathcal{E}}_1}\mathcal{P}[e^a a_a^\dagger \bar{S}_0^\dagger\boldsymbol{\chi}_0] &= \\ = (-2\alpha+\beta)\mathcal{P}_{\bar{\mathcal{E}}_1}\mathcal{P}[e^a y_a\boldsymbol{\chi}_0] &= \\ = -\frac{(s-n)}{l^2(d+2n-4)}\mathcal{P}[(s-n+1)e^a y_a\boldsymbol{\chi}_0 - e^a a_a^\dagger \bar{S}_0^\dagger\boldsymbol{\chi}_0]\,, &\end{aligned} \tag{C.28}$$

where we have re-expressed the coefficients in terms of n, s. Note that the projection to the subspace of elements annihilated by $S_0^\dagger$ is automatic in the last expression. Furthermore,

$$\mathcal{P}_{\bar{\mathcal{E}}_1}\mathcal{P}\left[-\frac{1}{l^2}e^a y_a(\frac{\partial}{\partial z}\boldsymbol{\chi})|_{z^\alpha=0}\right] = -\frac{s-n+1}{l^2}\mathcal{P}_{\bar{\mathcal{E}}_1}\mathcal{P}\left[e^a y_a\boldsymbol{\chi}_0\right] \tag{C.29}$$

and

$$\mathcal{P}_{\bar{\mathcal{E}}_1}\mathcal{P}\left[-\frac{1}{l^2}e^a a_a^\dagger(\frac{\partial}{\partial w}\boldsymbol{\chi})|_{z^\alpha=0}\right] = \frac{1}{l^2}\mathcal{P}_{\bar{\mathcal{E}}_1}\mathcal{P}\left[e^a a_a^\dagger \bar{S}_0^\dagger\boldsymbol{\chi}_0\right]. \tag{C.30}$$

Summing up all contribution one arrives at (4.60).

Finally, the structure of the last term in (4.58) is obvious if one observes that, for $\boldsymbol{\phi}_0 \in \Gamma(\bar{\boldsymbol{\mathcal{E}}}_{\mathbf{0}})$ and $\boldsymbol{\phi} = \mathcal{K}_2\boldsymbol{\phi}_0$, the projectors coincide: $\mathcal{P}_R^0\boldsymbol{\phi}_0 = \mathcal{P}_R\boldsymbol{\phi}$ and $\mathcal{K}_0\sigma\bar{\sigma}\mathcal{P}_R^0\boldsymbol{\phi}_0 = \sigma\bar{\sigma}\mathcal{P}_R^0\boldsymbol{\phi}_0$.

References

[1] M. Fierz and W. Pauli, *On relativistic wave equations for particles of arbitrary spin in an electromagnetic field*, *Proc. Roy. Soc. Lond.* **A173** (1939) 211.

[2] L.P.S. Singh and C.R. Hagen, *Lagrangian formulation for arbitrary spin. 1. The boson case*, *Phys. Rev.* **D 9** (1974) 898.

[3] C. Fronsdal, *Massless fields with integer spin*, *Phys. Rev.* **D 18** (1978) 3624.

[4] C. Fronsdal, *Singletons and massless, integral spin fields on de Sitter space (elementary particles in a curved space VII)*, *Phys. Rev.* **D 20** (1979) 848.

[5] W. Siegel, *Covariantly second quantized string*, *Phys. Lett.* **B 142** (1984) 276.

[6] W. Siegel, *Covariantly second quantized string, 2*, *Phys. Lett.* **B 149** (1984) 157 [*Phys. Lett.* **B 149** (1984) 157].

[7] W. Siegel, *Covariantly second quantized string. 3*, *Phys. Lett.* **B 149** (1984) 162 [*Phys. Lett.* **B 149** (1984) 162].

[8] W. Siegel and B. Zwiebach, *Gauge string fields*, *Nucl. Phys.* **B 263** (1986) 105.

[9] A. Neveu and P.C. West, *Gauge covariant local formulation of bosonic strings*, *Nucl. Phys.* **B 268** (1986) 125.

[10] A. Neveu, H. Nicolai and P.C. West, *Gauge covariant local formulation of free strings and superstrings*, *Nucl. Phys.* **B 264** (1986) 573.

[11] E. Witten, *Noncommutative geometry and string field theory*, *Nucl. Phys.* **B 268** (1986) 253.

[12] N. Ohta, *Covariant second quantization of superstrings*, *Phys. Rev. Lett.* **56** (1986) 440, erratum *ibid.* **56** (1986) 1316.

JHEP08(2006)013

[13] C.B. Thorn, *String field theory*, *Phys. Rept.* **175** (1989) 1.

[14] S. Ouvry and J. Stern, *Gauge fields of any spin and symmetry*, *Phys. Lett.* **B 177** (1986) 335.

[15] A.K.H. Bengtsson, *A unified action for higher spin gauge bosons from covariant string theory*, *Phys. Lett.* **B 182** (1986) 321.

[16] M. Henneaux and C. Teitelboim, *First and second quantized point particles of any spin*, ch. 9, pp. 113-152. Quantum mechanics of fundamental systems 2, Centro de Estudios Científicos de Santiago. Plenum Press, 1987.

[17] N. Bouatta, G. Compere and A. Sagnotti, *An introduction to free higher-spin fields*, `hep-th/0409068`.

[18] M.A. Vasiliev, *Nonlinear equations for symmetric massless higher spin fields in $(A)dS_d$*, *Phys. Lett.* **B 567** (2003) 139 [`hep-th/0304049`].

[19] M.A. Vasiliev, *Consistent equation for interacting gauge fields of all spins in (3+1)-dimensions*, *Phys. Lett.* **B 243** (1990) 378.

[20] M.A. Vasiliev, *Equations of motion of interacting massless fields of all spins as a free differential algebra*, *Phys. Lett.* **B 209** (1988) 491.

[21] M.A. Vasiliev, *Consistent equations for interacting massless fields of all spins in the first order in curvatures*, *Ann. Phys. (NY)* **190** (1989) 59.

[22] M.A. Vasiliev, *Conformal higher spin symmetries of 4d massless supermultiplets and* $\mathrm{OSp}(l, 2m)$ *invariant equations in generalized (super)space*, *Phys. Rev.* **D 66** (2002) 066006 [`hep-th/0106149`].

[23] K.B. Alkalaev, O.V. Shaynkman and M.A. Vasiliev, *On the frame-like formulation of mixed-symmetry massless fields in $(A)dS_d$*, *Nucl. Phys.* **B 692** (2004) 363 [`hep-th/0311164`].

[24] E. Sezgin and P. Sundell, *7d bosonic higher spin theory: symmetry algebra and linearized constraints*, *Nucl. Phys.* **B 634** (2002) 120 [`hep-th/0112100`].

[25] G. Barnich, M. Grigoriev, A. Semikhatov and I. Tipunin, *Parent field theory and unfolding in BRST first-quantized terms*, *Commun. Math. Phys.* **260** (2005) 147 [`hep-th/0406192`].

[26] M.A. Vasiliev, *Cubic interactions of bosonic higher spin gauge fields in AdS_5*, *Nucl. Phys.* **B 616** (2001) 106 [`hep-th/0106200`].

[27] S.W. MacDowell and F. Mansouri, *Unified geometric theory of gravity and supergravity*, *Phys. Rev. Lett.* **38** (1977) 739.

[28] K.S. Stelle and P.C. West, *Spontaneously broken de Sitter symmetry and the gravitational holonomy group*, *Phys. Rev.* **D 21** (1980) 1466.

[29] X. Bekaert, S. Cnockaert, C. Iazeolla and M.A. Vasiliev, *Nonlinear higher spin theories in various dimensions*, `hep-th/0503128`.

[30] B. Fedosov, *Deformation quantization and index theory*, Akademie-Verl., Berlin 1996.

[31] M. Bordemann, N. Neumaier and S. Waldmann, *Homogeneous Fedosov star products on cotangent bundles II: GNS representations, the WKB expansion and applications*, `q-alg/9711016`.

[32] I.A. Batalin, E.S. Fradkin and T.E. Fradkina, *Generalized canonical quantization of dynamical systems with constraints and curved phase space*, *Nucl. Phys.* **B 332** (1990) 723.

[33] M.A. Grigoriev and S.L. Lyakhovich, *Fedosov deformation quantization as a BRST theory*, *Commun. Math. Phys.* **218** (2001) 437 [hep-th/0003114].

[34] I.A. Batalin, M.A. Grigoriev and S.L. Lyakhovich, *Star product for second class constraint systems from a BRST theory*, *Theor. Math. Phys.* **128** (2001) 1109 [hep-th/0101089].

[35] A.K.H. Bengtsson, *BRST quantization in anti-de Sitter space and gauge fields*, *Nucl. Phys.* **B 333** (1990) 407.

[36] I.L. Buchbinder, A. Pashnev and M. Tsulaia, *Lagrangian formulation of the massless higher integer spin fields in the AdS background*, *Phys. Lett.* **B 523** (2001) 338 [hep-th/0109067].

[37] A. Sagnotti and M. Tsulaia, *On higher spins and the tensionless limit of string theory*, *Nucl. Phys.* **B 682** (2004) 83 [hep-th/0311257].

[38] V.E. Lopatin and M.A. Vasiliev, *Free massless bosonic fields of arbitrary spin in d-dimensional de Sitter space*, *Mod. Phys. Lett.* **A 3** (1988) 257.

[39] M.R. Gaberdiel and B. Zwiebach, *Tensor constructions of open string theories I: foundations*, *Nucl. Phys.* **B 505** (1997) 569 [hep-th/9705038].

[40] G. Barnich and M. Grigoriev, *Hamiltonian BRST and batalin-vilkovisky formalisms for second quantization of gauge theories*, *Commun. Math. Phys.* **254** (2005) 581 [hep-th/0310083].

[41] A. Dresse, P. Grégoire and M. Henneaux, *Path integral equivalence between the extended and nonextended hamiltonian formalisms*, *Phys. Lett.* **B 245** (1990) 192.

[42] C.B. Thorn, *Perturbation theory for quantized string fields*, *Nucl. Phys.* **B 287** (1987) 61.

[43] M. Bochicchio, *Gauge fixing for the field theory of the bosonic string*, *Phys. Lett.* **B 193** (1987) 31.

[44] M. Bochicchio, *String field theory in the Siegel gauge*, *Phys. Lett.* **B 188** (1987) 330.

[45] G. Barnich and M. Henneaux, *Isomorphisms between the Batalin-Vilkovisky antibracket and the Poisson bracket*, *J. Math. Phys.* **37** (1996) 5273 [hep-th/9601124].

[46] G. Bonelli, *On the covariant quantization of tensionless bosonic strings in AdS spacetime*, *JHEP* **11** (2003) 028 [hep-th/0309222].

[47] G. Barnich and M. Grigoriev, *BRST extension of the non-linear unfolded formalism*, hep-th/0504119.

[48] M.A. Vasiliev, *Unfolded representation for relativistic equations in (2+1) anti-de Sitter space*, *Class. and Quant. Grav.* **11** (1994) 649.

[49] O.V. Shaynkman and M.A. Vasiliev, *Scalar field in any dimension from the higher spin gauge theory perspective*, *Theor. Math. Phys.* **123** (2000) 683 [hep-th/0003123].

[50] G. Barnich, N. Bouatta and M. Grigoriev, *Surface charges and dynamical Killing tensors for higher spin gauge fields in constant curvature spaces*, *JHEP* **10** (2005) 010 [hep-th/0507138].

[51] J.-P. Schneiders, *An introduction to D-modules*, *Bull. Soc. Royale Sci. Liège,* **63(3-4)** (1994) 223.